Lecture 12: ϵ-expansion and Wilson-Fisher fixed point

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In this lecture, we see ...

- By applying the perturbative RG to GFP, we will find a new fixed point near the GFP. (Wilson-Fisher fixed point (WFFP))
- By replacing the GFP and the WFFP by their multi-component counterparts, we can obtain the ϵ-expansion of the universality classes of the XY model ($n = 2$) and of the Heisenberg model ($n = 3$).

Wilson-Fisher fixed point

- By inspecting the RG flow equation around GFP, we can obtain an $\epsilon(\equiv 4-d)$ dependent fixed point and its scaling properties to the first order in ϵ (ϵ-expansion).
- From this result one can obtain the lowest order approximation to the Wilson-Fisher fixed point, the fixed point that governs the Ising universality class in dimensions $2 < d < 4$.

RG flow equation around GFP

- Now, we are ready to actually compute the RG flow around the GFP searching for a new fixed point for the ϕ^4 model.
- Our tool is the RG flow equation around a fixed point.

$$\frac{dg_n}{d\lambda} = y_n g_n - \sum_{lm} c^n_{lm} g_l g_m + O(g^3) \quad (\lambda \equiv \log b) \tag{1}$$

- For the GFP, we already know

$$\phi_n \equiv [\![\phi^n]\!], \quad y_n = d - x_n, \quad x_n = nx = \frac{n}{2}(d-2)$$
$$c^n_{lm} \equiv \binom{l}{k}\binom{m}{k} k! \quad \left(k \equiv \frac{l+m-n}{2}\right) \tag{2}$$

The Z_2 symmetry

- Let us focus on the relevant fields at the GFP:

$$h \equiv g_1, \quad t \equiv g_2, \quad v \equiv g_3, \quad u \equiv g_4$$

- Note that (1) and (2) ensures that when we start with even fields only, odd fields are not generated by the RGT.
- In addition, we know that the critical point of the Ising model possesses the symmetry with respect to $S \leftrightarrow -S$.
- Therefore, we expect that the fixed point representing the Ising criticality should be found in the "even parity" manifold, i.e., $g_{2n+1} = 0$ $(h = v = 0)$.

ϵ-expansion

- In terms of the remaining fields, t and u, the flow equations are

$$\frac{dt}{d\lambda} = y_t t - c^t_{tt} t^2 - 2c^t_{tu} tu - c^t_{uu} u^2 + O(g^3) \tag{3}$$

$$\frac{du}{d\lambda} = y_u u - c^u_{tt} t^2 - 2c^u_{tu} tu - c^u_{uu} u^2 + O(g^3) \tag{4}$$

with $y_t = 2$ and $y_u = 4 - d \equiv \epsilon$.

- Hereafter, we regard ϵ as a small quantity.
- Let (t^*, u^*) be the non-trivial solution to the fixed-point equation, i.e., they are not zero and make the RHSs of (3) and (4) zero.
- By considering the order in ϵ, we see $t^* = O(\epsilon^2)$ and $u^* = O(\epsilon)$.

($\because$ By perturbation assuption, both u^* and t^* are small. Then, in (3), the only term that can possibely be the same order as t is u^2. Therefore, $t^* \sim u^{*2}$. With this in mind, inspecting (4) we see that ϵu must be comparable to u^2, so $u^* \sim O(\epsilon)$.)

General prescription with RG flow equation

Generally, the RG flow equation such as (3) and (4) takes the form $d\boldsymbol{g}/d\lambda = \boldsymbol{f}(\boldsymbol{g})$ where $\boldsymbol{f}(\cdot)$ is a function that maps p dimensional vector to another p dimensional vector. Once we obtain such a set of RG flow equations, we obtain the fixed point and its scaling properties as follows.

1. Find the solution to $\boldsymbol{f}(\boldsymbol{g}^*) = 0$. Then, $\mathcal{H}_{g^*}$ is the fixed point Hamiltonian.
2. Linearize the RG flow quation around g^*. Specifically, $d\Delta\boldsymbol{g}/d\lambda = Y\Delta\boldsymbol{g}$, where $\Delta\boldsymbol{g} \equiv \boldsymbol{g} - \boldsymbol{g}^*$ and $Y_{\mu\nu} \equiv df_\mu/dg_\nu|_{\boldsymbol{g}=\boldsymbol{g}^*}$ is the gradient matrix.
3. Diagonalize Y as $Y = P^{-1}\hat{Y}P$ with $\hat{Y}$ being a diagonal matrix whose μth diagonal element is y_μ.
4. Then, for the new set of parameters defined as $\boldsymbol{u} \equiv P\Delta\boldsymbol{g}$, we have $du_\mu/d\lambda = y_\mu u_\mu$, which means that u_μ is the new scaling field and y_μ is the corresponding scaling eigenvalue. (Accordingly, the new scaling operators can be defined as $\boldsymbol{\psi} \equiv (P^\mathsf{T})^{-1}\boldsymbol{\phi}$.)

Wilson-Fisher fixed point

- Now, keeping only the terms that can make difference in scaling dimensions in the $O(\epsilon)$ order, we obtain

$$\frac{dt}{d\lambda} = 2t - 96u^2 - 24tu \quad (\equiv A) \tag{5}$$

$$\frac{du}{d\lambda} = \epsilon u - 72u^2 - 16tu \quad (\equiv B) \tag{6}$$

$$\left(c^t_{uu} = \binom{4}{3}\binom{4}{3}3! = 96, \quad c^u_{uu} = \binom{4}{2}\binom{4}{2}2! = 72, \quad \text{etc.}\right)$$

- Then, the fixed point is

$$(t^*, u^*) = \left(\frac{\epsilon^2}{108}, \frac{\epsilon}{72}\right) \tag{7}$$

- We regard this as the lowest order approximation to the Wilson-Fisher fixed point (WFFP).

Linearization around the WFFP

- Following the general prescription, let us define $\Delta u \equiv u - u^*, \Delta t \equiv t - t^*$ and recast (5) and (6) in the form

$$\frac{d}{d\lambda}\begin{pmatrix}\Delta t\\ \Delta u\end{pmatrix} = Y\begin{pmatrix}\Delta t\\ \Delta u\end{pmatrix}.$$

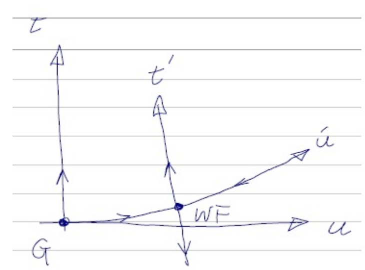
- The matrix Y can be obtained as

$$Y \equiv \begin{pmatrix}\frac{\partial A}{\partial t} & \frac{\partial A}{\partial u}\\ \frac{\partial B}{\partial t} & \frac{\partial B}{\partial u}\end{pmatrix}_{\Delta t = \Delta u = 0} = \begin{pmatrix}2 - 24u^* & -192u^*\\ -16u^* & \epsilon - 144u^*\end{pmatrix}$$
$$= \begin{pmatrix}2 - \frac{1}{3}\epsilon & -\frac{8}{3}\epsilon\\ -\frac{2}{9}\epsilon & -\epsilon\end{pmatrix}.$$

Scaling properties of the WFFP

- Thus, the linearized RG flow equation around the new fixed point is

$$\frac{d}{d\lambda}\begin{pmatrix}\Delta t\\ \Delta u\end{pmatrix} = \begin{pmatrix}2 - \frac{1}{3}\epsilon & -\frac{8}{3}\epsilon\\ -\frac{2}{9}\epsilon & -\epsilon\end{pmatrix}\begin{pmatrix}\Delta t\\ \Delta u\end{pmatrix}.$$



- Since the off-diagonal elements do not contribute to the eigenvalues to $O(\epsilon)$,

$$y_u^{\text{WF}} = -\epsilon \quad \text{and} \quad y_t^{\text{WF}} = 2 - \frac{\epsilon}{3}$$

- The "t-like" scaling field t' is relevant.

$$y_t^{\text{3DWF}} \approx 1.666\cdots \quad \left(y_t^{\text{3DIsing}} \approx 1.588(1)^*\right),$$
$$y_t^{\text{2DWF}} \approx 1.333\cdots \quad \left(y_t^{\text{2DIsing}} = 1\right)$$

* M. Hasenbusch, K. Pinn, and S. Vinti: PRB 59, 11471 (1999)

Scaling eigenvalue of h at WFFP

- The RG flow equation for h, which has been neglected so far, is

$$\frac{dh}{d\lambda} = y_h h - 2c^h_{th} th + (u^2 h\text{-term}) = \frac{d+2}{2} h - 4th + \cdots$$
$$\approx \left(\frac{d+2}{2} - 4t^* + \cdots \right) h$$

- Therefore, $y_h^{\mathrm{WF}} = \frac{d+2}{2} + O(\epsilon^2)$. Specifically,

  $y_h^{\mathrm{3D\ WF}} \approx 2.5$ and $y_h^{\mathrm{2D\ WF}} \approx 2.0.$

  The good agreements with

  $y^{\mathrm{3D\ Ising}} = 2.4817(4)^*$ and $y^{\mathrm{2D\ Ising}} = 1.875$ (exact).

  indicate the validity of the ϵ-expansion as well as the equivalence between the WFFPs and the Ising FPs.

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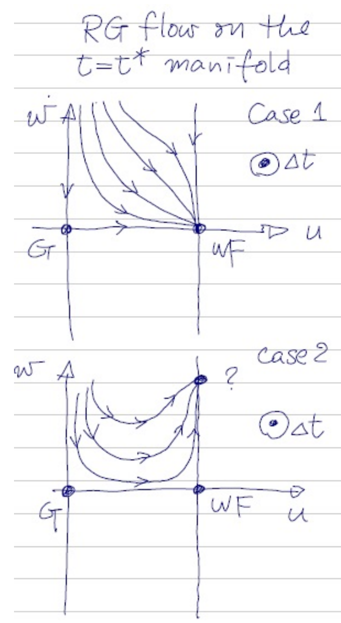
Irrelevancy of other operators



- Even if a field is irrelevant at the GFP, it may turn relevant at the WFFP. In such a case, the WFFP may not be the controlling FP. So, it is important to check whether there is no such fields.
- The RG flow equation for g_n around the GFP is

$$\frac{dg_n}{d\lambda} = \left(d - \frac{n}{2}(d-2) \right) g_n - 12n(n-1)ug_n,$$

- Remembering that $u^* = \epsilon/72$,

$$y_n^{\mathrm{WF}} = \left(d - \frac{n}{2}(d-2) \right) - 12n(n-1)\frac{4-d}{72}$$

- For $n \geq 6$, y_n^{WF} are negative:

$$y_n^{\mathrm{3DWF}} = \frac{18-2n-n^2}{6},\ y_n^{\mathrm{2DWF}} = \frac{6+n-n^2}{3}.$$

$O(n)$ models

- To apply the perturbative RG to the XY (O(2)) and the Heisenberg (O(3)) models we will introduce the multi-component ϕ^4 model.
- We can then construct the RG flow equation as before.

Multi-component ϕ^4 model

- Let us apply the perturbative RG to the XY (O(2)) or the Heisenberg (O(3)) models.
- To follow the same line of argument as before, we need something analogous to the ϕ^4 model to start with.
- So, let us consider multi-component field

$$\boldsymbol{\phi}(\boldsymbol{x}) \equiv (\phi_1(\boldsymbol{x}), \phi_2(\boldsymbol{x}), \cdots, \phi_n(\boldsymbol{x}))^{\mathsf{T}}$$

and the multi-component ϕ^4 model:

$$\mathcal{H} \equiv \int d\boldsymbol{x} \left(|\nabla \boldsymbol{\phi}|^2 + t\boldsymbol{\phi}^2 + u(\boldsymbol{\phi}^2)^2 - h\phi^1 \right)$$

- If $t = u = h = 0$, the n-components are independent and each represents a Gaussian fixed point. Therefore, it is a fixed point for the new Hamiltonian. (We call this fixed point the GFP, too.)

Correlation functions

- To get familiarized with the new model, let us consider $\langle \boldsymbol{\phi}^2(\boldsymbol{x})\boldsymbol{\phi}^2(\boldsymbol{y})\rangle_{\mathsf{GFP}}$.
- Since we can use Wick's theorem for the multi-component GFP,

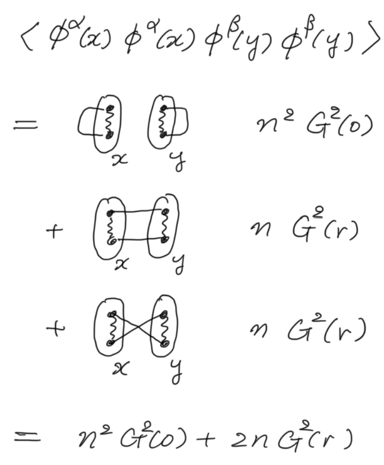
$$\begin{aligned}
&\langle \boldsymbol{\phi}^2(\boldsymbol{x})\boldsymbol{\phi}^2(\boldsymbol{y})\rangle \\
&\quad = \langle \phi_\alpha(\boldsymbol{x})\phi_\alpha(\boldsymbol{x})\phi_\beta(\boldsymbol{y})\phi_\beta(\boldsymbol{y})\rangle \quad \text{(Einstein's convention)} \\
&\quad = \langle \phi_\alpha(\boldsymbol{x})\phi_\alpha(\boldsymbol{x})\rangle\langle \phi_\beta(\boldsymbol{y})\phi_\beta(\boldsymbol{y})\rangle \\
&\qquad + 2\langle \phi_\alpha(\boldsymbol{x})\phi_\beta(\boldsymbol{y})\rangle\langle \phi_\alpha(\boldsymbol{x})\phi_\beta(\boldsymbol{y})\rangle \\
&\quad = n^2G^2(0) + 2nG^2(r)
\end{aligned}$$

where $r \equiv |\boldsymbol{x}-\boldsymbol{y}|$ and $G(r) \equiv \langle \phi_1(\boldsymbol{x})\phi_1(\boldsymbol{y})\rangle \approx r^{-2x}$ with $x = (d-2)/2$ as usual.

Diagrammatic representation

- We have seen that

$$\langle \boldsymbol{\phi}^2(\boldsymbol{x})\boldsymbol{\phi}^2(\boldsymbol{y})\rangle = n^2G^2(0) + 2nG^2(r)$$



- Compared with the previous case of $n = 1$, the difference is the factors n^2 and n.
- For a given pattern of Wick paring, draw the diagram like the one in the right with the correspondence:

$$\begin{aligned}
\text{wavy lines} &\leftrightarrow \begin{pmatrix}\text{repeated indices in} \\ \text{Einstein convention}\end{pmatrix} \\
\text{regular lines} &\leftrightarrow (\text{ Wick paring })
\end{aligned}$$

- To a diagram with g disconnected components, we assign the factor n^g.

Scaling operator φ_t (previously φ_2)

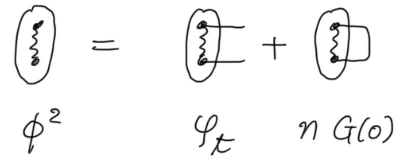
- As before, we can define the normal-order product, $[\![\cdots]\!]$, as the operator that we obtain after removing all contributions from the diagrams with inner connections.
- For example,

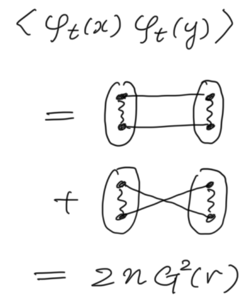
$$\varphi_t \equiv \left[\!\!\left[\boldsymbol{\phi}^2\right]\!\!\right] = \boldsymbol{\phi}^2 - nG^2(0)$$

- For the correlator of two ϕ_2s, we have

$$\langle \varphi_t(\boldsymbol{x})\varphi_t(\boldsymbol{y})\rangle = 2nG^2(r)$$

(See the diagram on the right.)





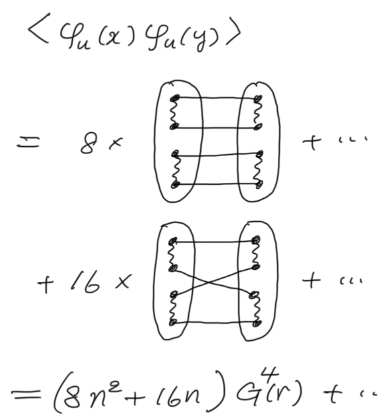
Scaling operator φ_u (previously φ_4)

- Similarly, we define ϕ_4 as

$$\varphi_u(\boldsymbol{x}) \equiv \left[\!\!\left[\left(\boldsymbol{\phi}^2(\boldsymbol{x})\right)^2\right]\!\!\right]$$

- Then, the correlator becomes

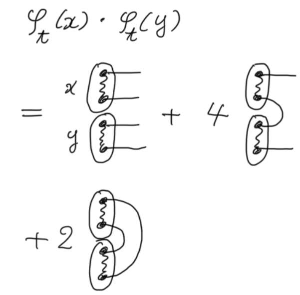
$$\begin{aligned}
&\langle \varphi_u(\boldsymbol{x})\varphi_u(\boldsymbol{y})\rangle \\
&\quad = \text{(Two-loop terms)} \\
&\qquad + \text{(One-loop terms)} \\
&\quad = 8n^2G^4(r) + 16nG^4(r) \\
&\quad = (8n^2 + 16n)G^4(r)
\end{aligned}$$



$c^u_{tt}, c^t_{tt}, c^u_{tu}, c^t_{tu}$ for $O(n)$ GFP

- First, let us expand $\varphi_t(\boldsymbol{x})\varphi_t(\boldsymbol{y})$.

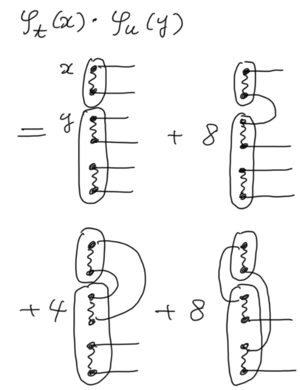
$$\varphi_t(\boldsymbol{x})\varphi_t(\boldsymbol{y}) \approx \varphi_u(\boldsymbol{x}) + 4G(r)\varphi_t(\boldsymbol{x}) + \cdots .$$



Thus, we obtain $c^u_{tt} = 1$ and $c^t_{tt} = 4$.

- For $\varphi_t(\boldsymbol{x})\varphi_u(\boldsymbol{y})$, we obtain

$$\begin{aligned}\varphi_t(\boldsymbol{x})\varphi_u(\boldsymbol{y}) &= \varphi_6(\boldsymbol{x}) + 8G(r)\varphi_u(\boldsymbol{x}) \\ &\quad + 4nG^2(r)\varphi_t(\boldsymbol{x}) + 8G^2(r)\varphi_t(\boldsymbol{x}) \\ &= \varphi_6 + 8G\varphi_u + (4n+8)G^2\varphi_t + \cdots .\end{aligned}$$



We obtain $c^u_{tu} = 8$ and $c^t_{tu} = 4(n+2)$.

Wilson-Fisher FP for $O(n)$ GFP

- The RG flow equation is

$$\begin{cases} \frac{dt}{d\lambda} = 2t - 32(n+2)u^2 - 8(n+2)tu & \equiv A \\ \frac{du}{d\lambda} = \epsilon u - 8(n+8)u^2 - 16tu & \equiv B \end{cases}$$

$$\Rightarrow \ (t^*, u^*) = \left(\frac{\epsilon^2}{4(n+8)^2}, \frac{\epsilon}{8(n+8)}\right)$$

- The flow equation for t around WFFP is

$$\frac{dt}{d\lambda} = (2 - 8(n+2)u^*)t \quad \Rightarrow \quad y_t^{\mathsf{WF}} = 2 - \frac{n+2}{n+8}\epsilon$$

- For h, we have

$$\frac{dh}{d\lambda} = (y_h^{\mathsf{G}} + O(\epsilon^2))h = \frac{d+2}{2}h$$

$$\Rightarrow \quad y_h^{\mathsf{WF}} = \frac{d+2}{2} = 3 - \frac{\epsilon}{2}$$

ϵ-expansion summary

		Ising ($n=1$)		XY ($n=2$)		Heisenberg ($n=3$)	
		ϵ-exp.	true	ϵ-exp.	true	ϵ-exp.	true
4D	y_t	2	2	2	2	2	2
	y_h	3	3	3	3	3	3
3D	y_t	1.67	1.59	1.60	1.49	1.55	1.41
	y_h	2.5	2.48	2.5	2.48	2.5	2.49
2D	y_t	1.33	1	1.20	—	1.09	—
	y_h	2.0	1.875	2.0	—	2.0	—

Summary

- By applying the perturbative RG to GFP, we have found a new fixed point near the GFP. (Wilson-Fisher fixed point (WFFP))
- We can apply the same perturbative argument to the n-component field ϕ, resulting in the ϵ-expansion of the universality classes of the XY model $(n = 2)$ and of the Heisenberg model $(n = 3)$. In 3D, the estimates of scaling dimesnions were surprisingly good, whereas even in 2D, they are not so far from the correct values.

Exercise 12.1: Obtain the OPE of $\varphi_u(\boldsymbol{x})\varphi_u(\boldsymbol{y})$ at the GFP, and show

$$c^u_{uu} = 8(n+8) \quad \text{and} \quad c^t_{uu} = 32(n+2).$$