Lecture 11: Perturbative Renormalization Group

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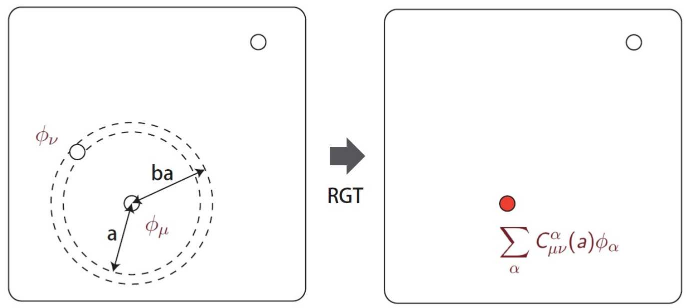
June 30, 2025

In this lecture, we see ...

- When there is a fixed point for which we know its OPE, we can derive, by a perturbative argument, a set of equations describing RG flow around it. (Then, we can study the behavior of other fixed points in its vicinity, as we will discuss in the next lecture.)
- We can obtain the renormalized Hamiltonian up to the 2nd order (or more if we try harder) in the case of GFP, which is the lowest non-trivial order.

General perturbative RG

- We decompose the field operator into the high-frequency component and the low-frequency component.
- Tracing out the high-fruquency component, followed by rescaling, yields the RG flow equations.
- In the RGT from the scale a to ab ($b = 1 + \lambda$), the product of two scaling operators within the distance of a, gives rise to new perturbative terms through OPE, which contributes non-linear terms in the RG flow equation.



Expanding the Hamiltonian around a fixed point

- Consider some fixed-point Hamiltonian, $\mathcal{H}_a^*$, with short-distant cut-off (lattice constant) a, and consider a general Hamiltonian expressed in terms of the scaling-operators at $\mathcal{H}_a^*$:

$$\mathcal{H}_a \equiv \mathcal{H}_a^* + V_a \qquad \left(V_a \equiv \sum_\alpha g_\alpha \int_a d\boldsymbol{x}\, \phi_\alpha(\boldsymbol{x}) \right)$$

where ϕ_α is the scaling operator at $\mathcal{H}^*$ with the dimension x_α.

$$\phi_\alpha(\boldsymbol{x}) \;\rightarrow\; \phi'_\alpha(\boldsymbol{x}') = \mathcal{R}_b \phi_\alpha(\boldsymbol{x}) = b^{x_\alpha} \phi_\alpha(\boldsymbol{x})$$

Outline of RGT for the expansion

- We carry out the general RGT program: partial trace and rescaling.
- We introduce the ultra-violet cut-off, a, which means: (i) When we expand $e^{-V_a(\phi)}$, the spatial integration like

$$\int_a d\boldsymbol{x}_1 d\boldsymbol{x}_2 \cdots d\boldsymbol{x}_n \; \phi_{\alpha_1}(\boldsymbol{x}_1)\phi_{\alpha_1}(\boldsymbol{x}_2)\cdots\phi_{\alpha_1}(\boldsymbol{x}_n)$$

  is restricted in the region where no two $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ are closer than a. (ii) The field ϕ_α contains only low-frequency component with $k < 1/a$.
- The partial trace will shift the cut-off a to $a' \equiv e^{\lambda} a \approx (1+\lambda)a$.
- The OPE is applied to every pair of operators that come within the mutual distance of a', and taking the summation with respect to the relative position of the two (This yields the factor $\Omega_d({a'}^d - a^d)$ $\approx \Omega_d d\lambda a^d$, where Ω_d is the volume of unit sphere.).

The partial trace

- We decompose the field operator as $\phi = \phi^\ell + \phi^s$ where ϕ^ℓ and ϕ^s are the long wave-length $(k < 1/a')$ and the short $(1/a' < k < 1/a)$ wave-length components of ϕ, respectively.
- In what follows, $\mathcal{H}_a (\equiv \mathcal{H}_a^* + V_a)$ is the perturbed Hamiltonian, $\mathcal{H}_a^*$ is the fixed point Hamiltonian, Z_s is the short wave-length contribution to the partition function, $\tilde{\mathcal{H}}_{a'}$ is the coarse-grained (but not yet re-scaled) perturbed Hamiltonian, and $\tilde{\mathcal{H}}_a^*$ is the coarse-grained fixed-point Hamiltonian. More specifically,

$$Z_s e^{-\tilde{\mathcal{H}}_{a'}^*(\phi^\ell)} \equiv \operatorname*{Tr}_{\{\phi^s\}} e^{-\mathcal{H}_a^*(\phi)}, \tag{1}$$

$$Z_s e^{-\tilde{\mathcal{H}}_{a'}(\phi^\ell)} \equiv \operatorname*{Tr}_{\{\phi^s\}} e^{-\mathcal{H}_a(\phi)}$$

$$= \operatorname*{Tr}_{\{\phi^s\}} \left\{ e^{-\mathcal{H}_a^*(\phi)} \left(1 - V_a(\phi) + \frac{1}{2}(V_a(\phi))^2 - \cdots \right) \right\}$$

The short wave-length average and the 1st order term

- The partial trace over ϕ^s goes like

$$\operatorname*{Tr}_{\{\phi^s\}} e^{-\mathcal{H}_a^*(\phi)-V_a(\phi)} = \operatorname*{Tr}_{\{\phi^s\}} e^{-\mathcal{H}_a^*(\phi)}\left(1 - V_a(\phi) + \frac{1}{2}V_a^2(\phi) + \cdots\right)$$
$$= Z_s e^{-\tilde{\mathcal{H}}_{a'}^*(\phi^\ell)}\left(1 - \langle V_a(\phi)\rangle_s + \frac{1}{2}\left\langle V_a^2(\phi)\right\rangle_s + \cdots\right) \quad (2)$$

where the short w.l. average is defined as

$$\langle\cdots\rangle_s \equiv \operatorname*{Tr}_{\{\phi^s\}} e^{-\mathcal{H}_a^*(\phi)}\cdots / \operatorname*{Tr}_{\{\phi^s\}} e^{-\mathcal{H}_a^*(\phi)}.$$

- The first order term is simply

$$\langle V_a(\phi)\rangle_s = \int_a d\boldsymbol{x}\sum_\alpha g_\alpha\langle\phi_\alpha(\boldsymbol{x})\rangle_s = \int_{a'} d\boldsymbol{x}\sum_\alpha g_\alpha\phi_a^\ell(\boldsymbol{x}) = V_{a'}(\phi^\ell)$$

The 2nd order term

- We can split the double integration into 2 parts:

$$\langle V_a^2(\phi)\rangle_s = \int_a d\boldsymbol{x}d\boldsymbol{y}\sum_{\alpha,\beta} g_\alpha g_\beta\langle\phi_\alpha(\boldsymbol{x})\phi_\beta(\boldsymbol{y})\rangle_s$$
$$= \sum_{\alpha,\beta} g_\alpha g_\beta\left(\int_{a'} + \int_{a<|\boldsymbol{x}-\boldsymbol{y}|<a'}\right) d\boldsymbol{x}d\boldsymbol{y}\langle\phi_\alpha(\boldsymbol{x})\phi_\beta(\boldsymbol{y})\rangle_s \quad (3)$$

- The first term is simply $(V_{a'}(\phi^\ell))^2$:

$$\sum_{\alpha,\beta} g_\alpha g_\beta\int_{a'} d\boldsymbol{x}d\boldsymbol{y}\langle\phi_\alpha(\boldsymbol{x})\phi_\beta(\boldsymbol{y})\rangle_s$$
$$= \sum_{\alpha,\beta} g_\alpha g_\beta\int_{a'} d\boldsymbol{x}d\boldsymbol{y}\,\phi_\alpha^\ell(\boldsymbol{x})\phi_\beta^\ell(\boldsymbol{y}) = (V_{a'}(\phi^\ell))^2$$

The "collision" term

- To conform the new cutoff a', the OPE must be applied to the second term in (3) representing operators too close to each other:

$$\begin{aligned}
&\sum_{\alpha,\beta} g_\alpha g_\beta \int_{a<|\boldsymbol{x}-\boldsymbol{y}|<a'} d\boldsymbol{x} d\boldsymbol{y} \, \langle \phi_\alpha(\boldsymbol{x})\phi_\beta(\boldsymbol{y})\rangle_s \\
&\quad = \sum_{\alpha,\beta} g_\alpha g_\beta \int_{a<|\boldsymbol{x}-\boldsymbol{y}|<a'} d\boldsymbol{x} d\boldsymbol{y} \sum_\mu \frac{c^\mu_{\alpha\beta}}{|\boldsymbol{x}-\boldsymbol{y}|^{x_\alpha+x_\beta-x_\mu}} \phi^\ell_\mu\left(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}\right) \\
&\quad = \sum_{\alpha,\beta} g_\alpha g_\beta \Omega_d((a')^d - a^d) \int_{a'} d\boldsymbol{x} \sum_\mu \frac{c^\mu_{\alpha\beta}}{a^{x_\alpha+x_\beta-x_\mu}} \phi^\ell_\mu(\boldsymbol{x}) \\
&\quad = \lambda \int_{a'} d\boldsymbol{x} \sum_\mu \left(\sum_{\alpha,\beta} c^\mu_{\alpha\beta} g_\alpha g_\beta (\Omega_d d a^{y_\alpha+y_\beta-y_\mu})\right) \phi^\ell_\mu(\boldsymbol{x})
\end{aligned}$$

The final form of the 2nd order term

Putting together, the 2nd order term in (3) becomes

$$\begin{aligned}
&\langle (V_a(\phi))^2\rangle_s \\
&= Z_s e^{-\tilde{\mathcal{H}}^*_{a'}} \left\{ (V_{a'}(\phi^\ell))^2 + \lambda \sum_{\mu,\alpha,\beta} \left(c^\mu_{\alpha\beta} g_\alpha g_\beta (\Omega_d d a^{y_\alpha+y_\beta-y_\mu})\right) \int_{a'} d\boldsymbol{x}\, \phi^\ell_\mu(\boldsymbol{x}) \right\} \\
&= Z_s e^{-\tilde{\mathcal{H}}^*_{a'}} \left((V_{a'}(\phi^\ell))^2 - 2 V^{(\text{int})}_{a'}(\phi^\ell) \right) \\
&V^{(\text{int})}_{a'}(\phi^\ell) \equiv -\frac{\lambda}{2} \sum_\mu \left(\sum_{\alpha,\beta} c^\mu_{\alpha\beta} g_\alpha g_\beta (\Omega_d d a^{y_\alpha+y_\beta-y_\mu})\right) \int_{a'} d\boldsymbol{x}\, \phi^\ell_\mu(\boldsymbol{x}).
\end{aligned}$$

Thus, the expansion (2) becomes

$$\operatorname*{Tr}_{\{\phi^s\}} e^{-\mathcal{H}^*_a(\phi) - V_a(\phi)} = Z_s e^{-\tilde{\mathcal{H}}^*_{a'}} \left(1 - V_{a'} + \frac{1}{2}(V_{a'})^2 - V^{(\text{int})}_{a'} + \cdots\right)$$

Summary of partial trace

- Finally, the partial trace results in

$$\operatorname*{Tr}_{\{\phi^s\}} e^{-\mathcal{H}_a^*(\phi)-V_a(\phi)} \approx Z_s e^{-\tilde{\mathcal{H}}_{a'}^*(\phi^\ell)-V_{a'}(\phi^\ell)-V_{a'}^{(\text{int})}(\phi^\ell)}$$

- Therefore, our Hamiltonian after the partial trace is

$$\begin{aligned}\tilde{\mathcal{H}}_{a'}(\phi^\ell) &= \tilde{\mathcal{H}}_{a'}^*(\phi^\ell) + V_{a'}(\phi^\ell) + V_{a'}^{(\text{int})}(\phi^\ell) \\ &= \tilde{\mathcal{H}}_{a'}^*(\phi^\ell) + \sum_\mu g_\mu \int_{a'} d\boldsymbol{x}\, \phi_\mu^\ell(\boldsymbol{x}) \\ &\quad - \frac{\lambda}{2}\sum_{\mu\alpha\beta} c_{\alpha\beta}^\mu g_\alpha g_\beta d\Omega_d a^{y_\alpha+y_\beta-y_\mu} \int_{a'} d\boldsymbol{x}\, \phi_\mu^\ell(\boldsymbol{x}) \\ &= \tilde{\mathcal{H}}_{a'}^*(\phi^\ell) + \sum_\mu \tilde{g}_\mu \int_{a'} d\boldsymbol{x}\, \phi_\mu^\ell(\boldsymbol{x})\end{aligned}$$

where $\tilde{g}_\mu \equiv g_\mu - \frac{\lambda}{2}\sum_{\mu\alpha\beta} c_{\alpha\beta}^\mu g_\alpha g_\beta d\Omega_d a^{y_\alpha+y_\beta-y_\mu}$

Rescaling and RG flow equation

- By re-scaling ($\boldsymbol{x}' \equiv b^{-1}\boldsymbol{x}$ and $\phi_\mu'(\boldsymbol{x}') \equiv b^{x_\mu}\phi_\mu^\ell(\boldsymbol{x})$),

$$\mathcal{H}_a'(\phi') = \tilde{\mathcal{H}}_{a'}(\phi^\ell) = \mathcal{H}_a^*(\phi') + \sum_\mu \tilde{g}_\mu \int_a d\boldsymbol{x}'\, b^{y_\mu}\phi_\mu'(\boldsymbol{x}')$$

$$\Rightarrow \quad g_\mu' = b^{y_\mu}\tilde{g}_\mu = b^{y_\mu}\left(g_\mu - \frac{\lambda}{2}\sum_{\alpha\beta} c_{\alpha\beta}^\mu g_\alpha g_\beta d\Omega_d a^{y_\alpha+y_\beta-y_\mu}\right).$$

- By absorbing the factor $\frac{d}{2}\Omega_d a^{y_\mu}$ in the definition of g_μ and g_μ',

$$g_\mu' = (1+\lambda)^{y_\mu} \times \left(g_\mu - \lambda\sum_{\alpha\beta} c_{\alpha\beta}^\mu g_\alpha g_\beta\right)$$

$$\frac{dg_\mu}{d\lambda} = y_\mu g_\mu - \sum_{\alpha\beta} c_{\alpha\beta}^\mu g_\alpha g_\beta + O(g^3)$$

Why can we apply OPE? I

To deal with the "collision" term, we used the OPE that is supposed to be only asymptotically correct, where the involving operators are viewed from another point far enough to them. So, can we justify the usage of the OPE in the perturbative RG? What view point should we assume "far enough"?

In the case of the collision term treatment, what we really need is the property that, under the coarse-grained Hamiltonian, the correlation function $\langle \psi_1(\boldsymbol{x})\psi_2(\boldsymbol{y})\rangle$ between two points far from each other shows the correct asymptotic behavior, i.e., the same asymptotic behavior as the Hamiltonian before the coarse-graining.

We cannot prove this equivalence in a mathematically rigorous fashion. As shown below, however, at least we can elucidate the assumption we are making here.

Why can we apply OPE? II

The correlation function can be expressed as

$$\langle \psi_1(\boldsymbol{x})\psi_2(\boldsymbol{y})\rangle = Z^{-1} \operatorname*{Tr}_{\{\phi^l\}} \left(e^{-\mathcal{H}^*_{a'}(\boldsymbol{\phi}) - V_{a'}(\phi^l) - V^{(\mathrm{int})}_{a'}(\phi^l)} \psi_1(\boldsymbol{x})\psi_2(\boldsymbol{y}) \right)$$

By expanding with respect to $V^{(\mathrm{int})}_{a'}$, we obtain

$$\langle \psi_1(\boldsymbol{x})\psi_2(\boldsymbol{y})\rangle \propto \operatorname*{Tr}_{\{\phi^l\}} \left(e^{-\mathcal{H}^* - V} \sum_{\boldsymbol{z}_1, \boldsymbol{z}_2, \cdots} \psi_1(\boldsymbol{x})\psi_2(\boldsymbol{y}) V^{(\mathrm{int})}(\boldsymbol{z}_1) V^{(\mathrm{int})}(\boldsymbol{z}_2) \cdots \right).$$

Consider that we restrict the summation $\sum_{\boldsymbol{z}_1, \boldsymbol{z}_2, \cdots}$ so that any two among $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}_1, \boldsymbol{z}_2, \cdots$ are "far enough", say the distance being more than a certain value R, to justify the use of the OPE. The assumption we made is that the asymptotic behavior of the correlation function $\langle \psi_1(\boldsymbol{x})\psi_2(\boldsymbol{y})\rangle$ in the $|\boldsymbol{x} - \boldsymbol{y}| \to \infty$ limit comes from this restricted summation. This is

Why can we apply OPE? III

natural because all the other contribution is the one from highly restricted configuration that at least one pair of points comes within the distance R, which effectively reduces the dimension of the spatial integration by d. Therefore, relative weight of such terms will become smaller as the distance $\boldsymbol{x} - \boldsymbol{y}$ gets larger and eventually zero in the infinite distance limit.

Perturbative RG around GFP

- The criticality of the Ising model in $d > 4$ is controled by the Gaussian fixed-point, though the critical behavior is modified by the dangerously irrelevant field.
- For $d < 4$, the Gaussian fixed-point is not stable w.r.t. the scaling operator ϕ_4. This motivates us to look for another fixed point by examining the perturbative RG flow around the Gaussian fixed point.

Critical property of the Ising model above 4-dimensions

- Consider the ϕ^4 model, with ϕ^2 and ϕ^4 terms. From the viewpoint of the perturbative RG around GFP, it is convenient to use the scaling fields ϕ_2 and ϕ_4, instead of ϕ^2 and ϕ^4:

$$\mathcal{H} = \int d\boldsymbol{x} \left(|\nabla\phi|^2 + t\phi_2 + u\phi_4 - h\phi \right)$$

- The scaling eigenvalues for these terms are

$$x_2 = 2x = d - 2 \;\Rightarrow\; y_2 = d - x_2 = 2$$
$$x_4 = 4x = 2(d-2) \;\Rightarrow\; y_4 = d - x_4 = 4 - d.$$

- Since ϕ_4 is irrelevant if $d > 4$, the critical behavior of the ϕ^4 model at $t = 0$ (and therefore the Ising model at $T = T_c$ as well) is described by the GFP.

Dangerous irrelevant operator for $d > 4$

- According to the general argument (see Lecture 7), the spontaneous magnetization should scale like

$$m \sim L^{-d+y_h} = L^{-x_h} \sim t^{\frac{x_h}{y_t}} = t^{\frac{d-2}{4}}. \quad \text{(wrong)}$$

- However, we saw that the mean-field theory correctly describes the critical behavior for $d > 4$ (Ginzburg criterion), which means that

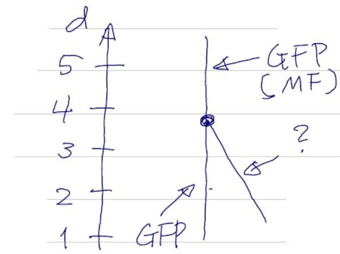
$$m \sim t^{\frac{1}{2}}. \quad \text{(correct)}$$

- This apparent contradiction comes from the nature of the irrelevant field u. Specifically, since the ϕ^4 model at or below the critical point ($t \leq 0$) is not well-defined when $u = 0$, we cannot simply put $u = 0$ in the scaling form as we did in the general argument.

Perturbative RG around GFP

- We have derived the general RG flow equation around a fixed-point.

$$\frac{dg_\mu}{d\lambda} = y_\mu g_\mu - \sum_{\alpha\beta} c^\mu_{\alpha\beta} g_\alpha g_\beta \qquad (4)$$



- If we apply this to GFP, we immediately notice that, in $d > 4$, there is no relevant field other than t, implying that the GFP governs the critical phenomena of the ϕ^4 model.
- Even below four dimensions, we may be able to obtain a new fixed point from (4) if it is near the GFP.
- In other words, we may try to find g_μ that makes the r.h.s. of (4) zero and deduce its properties from (4). (Next lexture)

Summary

- We have derived a set of equations describing RG flow around a given fixed point.
- We can obtain the renormalized Hamiltonian up to the 2nd order (or more if we try harder) in the case of GFP, which is the lowest non-trivial order.
- Above four dimensions, the critical point is controled by the Gaussian fixed point.
- However, the dangerously irrelevant field, u, modifies the critical behaviors to mean-field like.
- Below four dimensions, the critical point is not controled by the Gaussian fixed point because u becomes relevant.
- We may be able to find the "true" fixed point by analyzing the RG flow equation. (Next lecture)

Exercise 11.1: We saw an apparent contradiction between the general scaling argument and the mean-field behaviors expected from the Ginzburg criterion. Think of a scaling form of the singular part of the free energy that obeys the scaling properties expected from the general argument, and, at the same time, produces the correct mean-field critical behaviors.