Lecture 10: Operator Product Expansion

Naoki KAWASHIMA

ISSP, U. Tokyo

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In this lecture, we see ...

- Product of two scaling operators can be expanded in terms of scaling operators. (OPE)
- Such an expansion determines the RG-flow structure around the fixed point (will be discussed in the next lecture).
- For the Gaussian fixed point, all the coefficients of the OPE can be exactly obtained.

General Framework

- Product of two scaling operators, defined at some distance from each other, can be expanded as a linear combination of scaling operators.
- Considering the 3-point correlators and taking into account of the scaling properties of operators, the general form of the OPE coefficients can be fixed up to universal constants.

Product of two is expandable

- Previously, we introduced scaling operators $\{\varphi_\mu\}$ as something that spans the space of local operators:

$$\forall Q(\boldsymbol{x}) \exists q_\mu \left(Q(\boldsymbol{x}) = \sum_\mu q_\mu \varphi_\mu(\boldsymbol{x}) \right) \tag{1}$$

- The RGT's action on φ_μ is $\varphi'_\mu(\boldsymbol{x}') = b^{x_\mu} \varphi_\mu(\boldsymbol{x}),$ which leads to

$$\langle \varphi_\mu(\boldsymbol{x}) \varphi_\nu(\boldsymbol{y}) \rangle \sim |\boldsymbol{x} - \boldsymbol{y}|^{-2x_\mu}$$

- Let us consider the product of two scaling operators, $\varphi_\mu(\boldsymbol{x})\varphi_\nu(\boldsymbol{y})$. This product must appear to be a "local" operator when we view it from a point $\boldsymbol{z}$ far away from $\boldsymbol{x}$ and $\boldsymbol{y}$ (i.e., $|\boldsymbol{z} - \boldsymbol{x}| \gg |\boldsymbol{y} - \boldsymbol{x}|$).
- Then, we should be able to expand it:

$$\varphi_\mu(\boldsymbol{x})\varphi_\nu(\boldsymbol{y}) = \sum_\alpha C^\alpha_{\mu\nu}(\boldsymbol{x} - \boldsymbol{y}) \varphi_\alpha \left(\frac{\boldsymbol{x} + \boldsymbol{y}}{2} \right)$$

(Equality holds only asymptotically.)

Three-point correlator (1/2)

- Let us consider three-point correlation function:

$$G_{\mu\nu\lambda}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \equiv \langle \varphi_\mu(\boldsymbol{x})\varphi_\nu(\boldsymbol{y})\varphi_\lambda(\boldsymbol{z})\rangle$$

- By applying the RGT and then expanding $\varphi_\mu\varphi_\nu$, we have

$$\begin{aligned} G_{\mu\nu\lambda}(\boldsymbol{x}',\boldsymbol{y}',\boldsymbol{z}') &= b^{x_\mu+x_\nu+x_\lambda} G_{\mu\nu\lambda}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \\ &= b^{x_\lambda+x_\mu+x_\nu} \sum_\alpha C^\alpha_{\mu\nu}(\boldsymbol{x}-\boldsymbol{y}) G_{\alpha\lambda}\left(\frac{\boldsymbol{x}+\boldsymbol{y}}{2},\boldsymbol{z}\right) \end{aligned} \tag{2}$$

- By reversing the order of the operations, we have

$$\begin{aligned} G_{\mu\nu\lambda}(\boldsymbol{x}',\boldsymbol{y}',\boldsymbol{z}') &= \sum_\alpha C^\alpha_{\mu\nu}(\boldsymbol{x}'-\boldsymbol{y}') G_{\alpha\lambda}\left(\frac{\boldsymbol{x}'+\boldsymbol{y}'}{2},\boldsymbol{z}'\right) \\ &= \sum_\alpha C^\alpha_{\mu\nu}(\boldsymbol{x}'-\boldsymbol{y}') b^{x_\alpha+x_\lambda} G_{\alpha\lambda}\left(\frac{\boldsymbol{x}+\boldsymbol{y}}{2},\boldsymbol{z}\right) \end{aligned} \tag{3}$$

Three-point correlator (2/2)

- Comparing (2) and (3), we conclude

$$C^\alpha_{\mu\nu}(\boldsymbol{r}) = b^{-x_\mu-x_\nu+x_\alpha} C^\alpha_{\mu\nu}\left(\frac{\boldsymbol{r}}{b}\right).$$

This leads to

$$\exists c^\alpha_{\mu\nu}\left(C^\alpha_{\mu\nu}(\boldsymbol{r}) = \frac{c^\alpha_{\mu\nu}}{r^{x_\mu+x_\nu-x_\lambda}}\right).$$

Therefore, we have the operator-product expansion:

$$\varphi_\mu(\boldsymbol{x})\varphi_\nu(\boldsymbol{y}) = \sum_\alpha \frac{c^\alpha_{\mu\nu}}{r^{x_\mu+x_\nu-x_\lambda}} \varphi_\alpha(\boldsymbol{x}) \qquad \text{(OPE)}$$

Universality

- By normalizing the scaling operators, φ_μ, so that

$$\lim_{|\boldsymbol{x}-\boldsymbol{y}|\to\infty} |\boldsymbol{x}-\boldsymbol{y}|^{2x_\mu}\langle\varphi_\mu(\boldsymbol{x})\varphi_\mu(\boldsymbol{y})\rangle = 1$$

  the OPE coefficient $c_{\mu\nu}^{\alpha}$ can be fixed. The resulting coefficients are universal numbers.

- We assume that thus fixed OPE coefficients $c_{\mu\nu}^{\alpha}$ are universal, and characterizing property of the fixed-point, together with the scaling dimensions, x_μ. In other words, they do not depend on the details of the system, but depend only on the symmetry, the space dimension, etc. (This assumption of universality is similar to the assumption of very existence of the fixed-point of the RGT. It is at least supported by several exactly solvable cases.)

OPE at the Gaussian fixed point

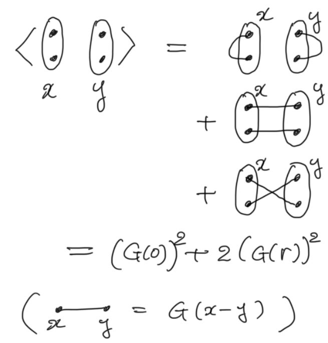
- The scaling operators at the Gaussian fixed point can be obtained through the normal order product: $\varphi_n \equiv [\![\phi^n]\!]$.

- We can compute exact OPE coefficients of the gaussian fixed point.

A hint for scaling operators — Wick's theorem

- Consider the operator $(\phi(\boldsymbol{x}))^2$ and its correlation function.

$$\langle \phi^2(\boldsymbol{x})\phi^2(\boldsymbol{y})\rangle = \langle \underset{1}{\phi(\boldsymbol{x})}\,\underset{2}{\phi(\boldsymbol{x})}\,\underset{3}{\phi(\boldsymbol{y})}\,\underset{4}{\phi(\boldsymbol{y})}\rangle$$
$$= \langle \underset{1}{\phi(\boldsymbol{x})}\,\underset{2}{\phi(\boldsymbol{x})}\rangle\langle \underset{3}{\phi(\boldsymbol{y})}\,\underset{4}{\phi(\boldsymbol{y})}\rangle$$
$$+ \langle \underset{1}{\phi(\boldsymbol{x})}\,\underset{3}{\phi(\boldsymbol{y})}\rangle\langle \underset{2}{\phi(\boldsymbol{x})}\,\underset{4}{\phi(\boldsymbol{y})}\rangle$$
$$+ \langle \underset{1}{\phi(\boldsymbol{x})}\,\underset{4}{\phi(\boldsymbol{y})}\rangle\langle \underset{2}{\phi(\boldsymbol{x})}\,\underset{3}{\phi(\boldsymbol{y})}\rangle$$
$$= G^2(0) + 2G^2(r) \qquad (4)$$



where $r \equiv |\boldsymbol{x}-\boldsymbol{y}|$ and

$$G(r) \sim \frac{1}{r^{2x}} \quad \text{where} \quad x \equiv \frac{d-2}{2}$$

($x \equiv$ The scaling dim. of $\phi_1 = \phi$)

Scaling operators

What are scaling operators at the Gaussian fixed point?

- If the constant term $G(0)^2$ in $\langle \phi^2(\boldsymbol{x})\phi^2(\boldsymbol{y})\rangle = G(0)^2 + 2G(r)^2$ were absent, $\phi^2(\boldsymbol{x})$ would be regarded as a scaling operator.
- This observation leads us to define

$$\varphi_2(\boldsymbol{x}) \equiv \phi(\boldsymbol{x})^2 - \langle \phi(\boldsymbol{x})^2\rangle,$$

which has the "pure" two-point correlator

$$\langle \varphi_2(\boldsymbol{x})\varphi_2(\boldsymbol{y})\rangle = \langle (\phi^2(\boldsymbol{x}) - G(0))(\phi^2(\boldsymbol{y}) - G(0))\rangle$$
$$= \langle \phi^2(\boldsymbol{x})\phi^2(\boldsymbol{y})\rangle - G(0)^2 = 2G(r)^2 = \frac{2}{r^{4x}}$$

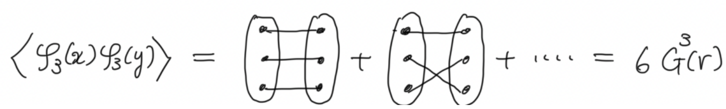
- Therefore, φ_2 is the scaling operator with the dimension $x_2 \equiv 2x$.

Normal-ordered operator

- The key to finding general scaling operators is to eliminate the diagrams with "internal connections" such as the first term in (4).
- Therefore, it would be convenient to introduce a symbol $[\![\cdots]\!]$ as

$$[\![A(\boldsymbol{x})]\!] \equiv A(\boldsymbol{x}) - \begin{pmatrix}\text{All terms represented by} \\ \text{diagrams with internal} \\ \text{connections}\end{pmatrix}$$

  The operator thus defined is called "normal-ordered."
- When considering correlations among normal-ordered operators, by definition, we can forget about the internal lines. Therefore, 2-point correlators do not have constant terms, which makes the normal-ordered operator $\varphi_n \equiv [\![\phi^n]\!]$ a scaling operator. For example,



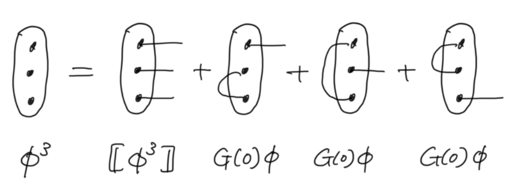
Scaling operators $\varphi_1 \equiv [\![\phi]\!]$, $\varphi_2 \equiv [\![\phi^2]\!]$ and $\varphi_3 \equiv [\![\phi^3]\!]$

- Obviously, $\varphi_1 \equiv [\![\phi]\!] = \phi$.
- For ϕ^2, as we have seen already

$$\phi^2 = [\![\phi^2]\!] + \langle \phi^2 \rangle \;\Rightarrow\; \varphi_2 \equiv \phi^2 - \langle \phi^2 \rangle$$

- For ϕ^3, from the diagram below, we obtain

$$\phi^3 = [\![\phi^3]\!] + 3\langle \phi^2 \rangle \phi$$



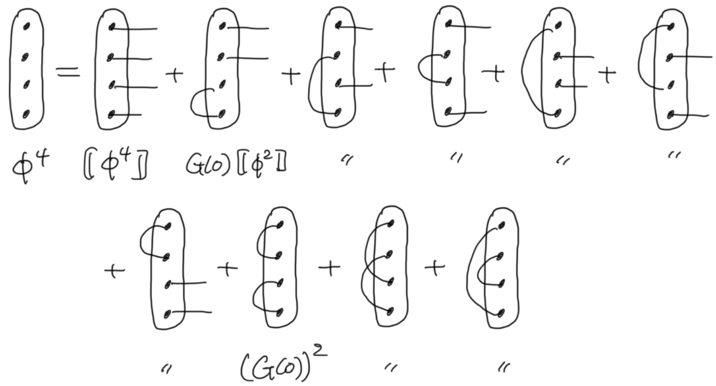
  Therefore,

$$\varphi_3 \equiv [\![\phi^3]\!] = \phi^3 - 3G(0)\phi$$

Scaling operator $\varphi_4 \equiv [\![\phi^4]\!]$

- By explicitly writing down the diagrams contributing to ϕ^4,

$$\phi^4 = [\![\phi^4]\!] + 6\langle\phi^2\rangle[\![\phi^2]\!] + 3\langle\phi^2\rangle^2$$



Therefore,

$$\varphi_4 = [\![\phi^4]\!] = \phi^4 - 6G(0)\varphi_2 - 3G(0)^2$$
$$= \phi^4 - 6G(0)\phi^2 + 3G(0)^2$$

Scaling operators of Gaussian fixed-point

- To summarize, the scaling operators of Gaussian fixed point are

$$\varphi_n \equiv [\![\phi^n]\!]$$

- For the standard normalization, consider

$$\langle\varphi_n(\boldsymbol{x})\varphi_n(\boldsymbol{y})\rangle = \sum_{\substack{\text{all pairing}\\ \text{patterns}}} \langle\phi(\boldsymbol{x})\phi(\boldsymbol{y})\rangle^n = n!G^n(r) = \frac{n!}{r^{2nx}}.$$

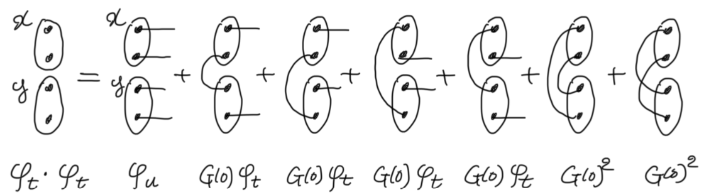
Therefore, $\hat{\varphi}_n \equiv \frac{1}{\sqrt{n!}}\varphi_n$ is the normalized scaling operator.

- The scaling dimension is obviously $x_n \equiv nx = \frac{n}{2}(d-2)$.
- Because of the "no internal line" condition, the scaling operators are orthogonal to each other

$$\langle\hat{\varphi}_m(\boldsymbol{x})\hat{\varphi}_n(\boldsymbol{y})\rangle = \frac{\delta_{mn}}{r^{2x_m}}.$$

Expansion of $\varphi_2(\boldsymbol{x})\varphi_2(\boldsymbol{y})$

- Consider the product of two operators $\varphi_2(\boldsymbol{x})\varphi_2(\boldsymbol{y})$.



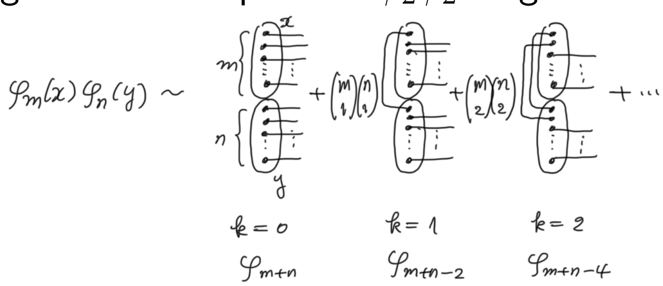
From this diagram, we obtain

$$\varphi_2(\boldsymbol{x})\varphi_2(\boldsymbol{y}) \sim \varphi_4\left(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}\right) + 4G(r)\varphi_2\left(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}\right) + 2G^2(r),$$

or more symbolically, $\varphi_2 \cdot \varphi_2 \sim \varphi_4 + 4\varphi_2 + 2.$

OPE of Gaussian fixed-point

- We can generalize the product $\phi_2\phi_2$ to general two operators.



$$\varphi_m(\boldsymbol{x})\varphi_n(\boldsymbol{y}) \sim \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} k! G^k(r)\,\varphi_{m+n-2k}\left(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}\right)$$

$$= \sum_{l=|m-n|,|m-n|+2,\cdots,m+n} \frac{c_{mn}^l}{|\boldsymbol{x}-\boldsymbol{y}|^{x_m+x_n-x_l}}\,\varphi_l\left(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}\right)$$

$$\left(c_{mn}^l = \binom{m}{k}\binom{n}{k}k! \quad \left(k \equiv \frac{m+n-l}{2}\right)\right)$$

Summary

- Generally, we can expand the product of the two scaling operators in terms of scaling operators (OPE), which takes the form

$$\varphi_m(\boldsymbol{x})\varphi_n(\boldsymbol{y}) = \sum_l \frac{c_{mn}^l}{|\boldsymbol{x}-\boldsymbol{y}|^{x_m+x_n-x_l}}\varphi_l\left(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}\right).$$

- The constants c_{mn}^l are universal quantities (provided that the scaling operators are properly normalized).
- For the Gaussian model, the scaling operator can be explicitly defined by the normal-ordering as $\varphi_n \equiv [\![\phi^n]\!]$, and its scaling dimension is $x_n \equiv nx = n(d-2)/2$.
- The OPE at the Gaussian fixed-point is characterized by

$$c_{mn}^l = \binom{m}{k}\binom{n}{k}k! \quad \left(k \equiv \frac{m+n-l}{2}\right)$$

Exercise 10.1: For the Gaussian model, obtain φ_5 in terms of ϕ^k $(k = 1, 2, 3, \cdots)$, following the same argument as in the lecture.