Lecture 6: Simple Real-Space RG Transformations

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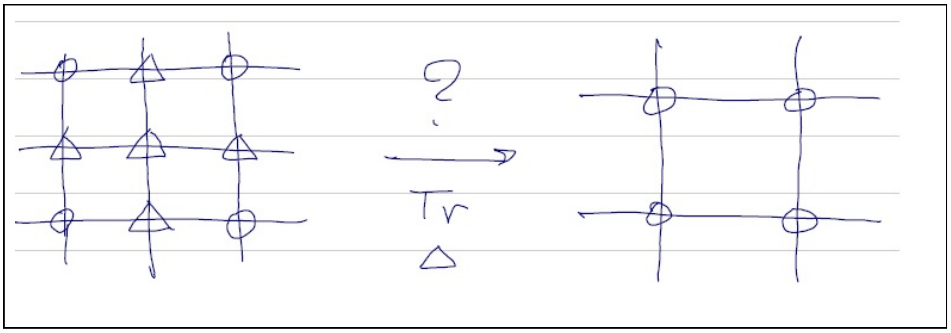
May 26 & June 2, 2025

In this lecture, we see

- The fixed point of the RG transformation is important in understanding our world.
- Real-space renormalization group transformation is generally impossible to carry out in dimension higher than 1. Therefore, it requires some approximation.
- A decimation-based RG can be approximately done by a method proposed by Migdal and Kadanoff.
- Generally, the fixed point of RG transformation (RGT) represents the critical point.
- MKRG produces a non-trivial evaluates of critical exponents.
- However, they do not generally agree with the correct values, and it is not obvious how to systematically improve the approximation.

1D was easy. Can we do the same in 2D case?



RG is trickier for $d > 1$

- Consider, for example, coarse-graining by decimation.

$$\tilde{S}_{\boldsymbol{x}} \equiv S_{\boldsymbol{x}} \quad \text{for } \boldsymbol{x} \in \Omega' \equiv \{(2ma, 2na)|m, n = 0, 1, 2, \cdots, L/2\}$$

- The partial trace can be taken (at least formally) as

$$e^{-\tilde{\mathcal{H}}_{2a}(\tilde{\boldsymbol{S}},\tilde{K})} \equiv \operatorname*{Tr}_{\{S_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\Omega\setminus\Omega'}} e^{-\mathcal{H}_a(\boldsymbol{S},K)}$$

- There are paths that connect two remaining spins, say $S_{\boldsymbol{r}}$ and $S_{\boldsymbol{r}'}$ ($\boldsymbol{r}, \boldsymbol{r}' \in \Omega'$), through $\Omega\setminus\Omega'$. Tracing out the spins along the path give rise to the long-range interaction between $S_{\boldsymbol{r}}$ and $S_{\boldsymbol{r}'}$.
- There are n-body ($n > 2$) interactions among the remaining spins, because, for example, $\sum_{S_0}(1 + tS_0S_1)(1 + tS_0S_2)(1 + tS_0S_3)$ $(1 + tS_0S_4)$ contains the term like $t^4S_1S_2S_3S_4$.
- As a result, unlike the 1D case, the renormalized Hamiltonian is too complicated to deal with.

$d > 1$ needs approximation

In 1D, the RG map transforms the simple Hamiltonian with only 2 parameters, the magnetic field H and the nearest neighbor interaction K, $\mathcal{H} \equiv -H \sum_i S_i - K \sum_{(ij)} S_i S_j$ into

$$\mathcal{H}' \equiv -H' \sum_i S_i' - K' \sum_{(ij)} S_i' S_j',$$

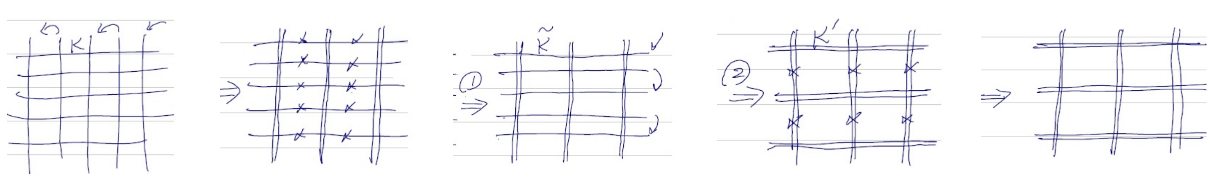
the exact same form with modified coupling constants K' and H'.

However, in 2D, the original simple Hamiltonian would be transformed into

$$\mathcal{H}' = -\sum_i H' S_i' - \sum_{ij} K_{ij}' S_i' S_j' - \sum_{ijk} K_{ijk}'' S_i' S_j' S_k' - \cdots$$

We can carry out the RG mapping only approximately.

Migdal-Kadanoff RG for 2D Ising model



1. Bunch up two vertical lines.
2. Partial trace of spins ($\times$) on horizontal bonds. (①: $\operatorname{th} \tilde{K} = \operatorname{th}^2 K$)
3. Bunch up two horizontal lines. (②: $K' = 2\tilde{K}$)
4. Partial trace of intermediate spins ($\times$) on vertical bonds.
5. Trivial re-scaling (just replace r' by r'/b and do nothing on K').

simple Migdal-Kadanoff

$$t' = \operatorname{th}(2\operatorname{ath}(t^2)) \left(= \frac{2t^2}{1+t^4} \right) \quad (t \equiv \operatorname{th} K,\ t' \equiv \operatorname{th} K')$$

Effect of RGT on correlation function

- Consider a RG transformation with scaling factor b

$$t' = R_b(t),$$

e.g., $R_2(t) = \mathrm{th}(2\,\mathrm{ath}(t^2))$ for MKRG.

- Obviously, the correlation length of the renormalized system ξ' should be equal to ξ/b. At the same time, ξ and ξ' are the values of the same function at different arguments, i.e., $\xi = \xi(t)$ and $\xi' = \xi(t')$. Therefore, the correlation as a function of the coupling constant must satisfy

$$\xi(t') = b^{-1}\xi(t). \tag{1}$$

RG fixed point and $y_t = 1/\nu$

- The RG fixed-point t_c is defined by $t_c = R_b(t_c)$.
- The RGT amplifies the 'deviation' from the fixed-point as

$$\delta t \to \delta t' = t' - t_c = R_b(t_c + \delta t) - t_c \approx R_b'(t_c)\delta t$$

- Therefore, (1) means $\xi(t_c + R_b'\delta t) \approx b^{-1}\xi(t_c + \delta t)$.
- Since the exponent ν is defined by $\xi \propto (\delta t)^{-\nu}$,

$$(R_b'\delta t)^{-\nu} = b^{-1}(\delta t)^{-\nu} \quad \to \quad R_b'^{-\nu} = b^{-1}$$

$$\to \quad y_t \equiv \frac{1}{\nu} = \frac{\log R_b'(t_c)}{\log b} \tag{2}$$

Derivatives of RG transformation are critical exponents.

RG fixed point and $y_t = 1/\nu$ (numerical estimates)

- For the Migdal-Kadanoff RGT for 2D Ising model, we have

$$t_c = R_2(t_c) = \frac{2t_c^2}{1+t_c^4} \to \; t_c = 0.54368\cdots$$

(cf: $t_c^{\text{exact}} = \sqrt{2}-1 = 0.4142\cdots$)

- With some arithmetic, we can get

$$R_2'(t_c) = \frac{2(1-t_c)}{t_c} \approx 1.676$$
$$\to y_t \equiv 1/\nu \approx \log 1.676/\log 2 \approx 0.747$$

(cf: $y_t^{\text{exact}} = 1$, $y_t^{\text{mean field}} = 2$)

Not bad, but ad-hoc (not obvious how to improve).

Infinitesimal MKRG

- The MKRG mapping with the scaling factor b can be summarized as

$$t' = R_2(t) \equiv \text{th}(2\,\text{ath}(t^2)).$$

- This can be generalized formally to general integer $b > 1$ as

$$t' = R_b(t) \equiv \text{th}(b\,\text{ath}(t^b)).$$

  Although the corresponding operation cannot be defined for non-integer b, let us assume that it is still meaningful.
- After all, "bunching-up" two lines to one by one step might be too crude. It may become less harmful if we bunch-up as small number of lines as possible, i.e., taking $b = 1 + \lambda$ where $0 < \lambda \ll 1$

Does this infinitesimal RG still yield sensible results?

Infinitesimal RG (general argument)

- In general, suppose some RG transformation $t' = R_b(t)$ with continuous scaling factor $b = 1 + \lambda$.
- Using the notation $\dot{f} \equiv \partial f/\partial b$, the critical point is determined by

$$t_c = R_{1+\lambda}(t_c) = R_1(t_c) + \lambda \dot{R}_1(t_c) \Rightarrow \dot{R}_1(t_c) = 0.$$

($\dot{R}_1(t)$ is called "beta function" and the symbol $\beta(t)$ is often used.)

- Using $R_1(t) = t$ (therefore $R_1'(t) = 1$, and $R_1''(t) = 0$), in the lowest order in λ, the scaling dimension y_t can be obtained by

$$y_t(1+\lambda) = \frac{\log\left(R'_{1+\lambda}(t_c(1+\lambda))\right)}{\log(1+\lambda)} \quad \text{(See Eq.(2))}$$

$$\approx \frac{1}{\lambda}\log\Big(R_1'(t_c(1)) + R_1''(t_c(1))\Delta t_c + \dot{R}_1'(t_c(1))\lambda\Big) \approx \dot{R}_1'(t_c(1)).$$

$\dot{R}_1(t_c) = 0$ and $y_t = \dot{R}_1'(t_c)$

Infinitesimal MKRG (numerical estimates)

- For $b = 1 + \lambda$ ($\lambda \ll 1$), defining $t \equiv \operatorname{th} K$, we obtain

$$t' = R_b(t) = \operatorname{th}(b \operatorname{ath} t^b) \approx t + \lambda \dot{R}_1(t)$$

$$\dot{R}_1(t) \equiv \left.\frac{\partial R_b(t)}{\partial b}\right|_{b\to 1} = (1-t^2)\operatorname{ath} t + t\log t$$

- The critical point $t = t_c$ is determined by $\dot{R}_1(t_c) = 0$, which yields

$$t_c = \sqrt{2} - 1 \quad \text{(Exactly agrees with the correct value!)}$$

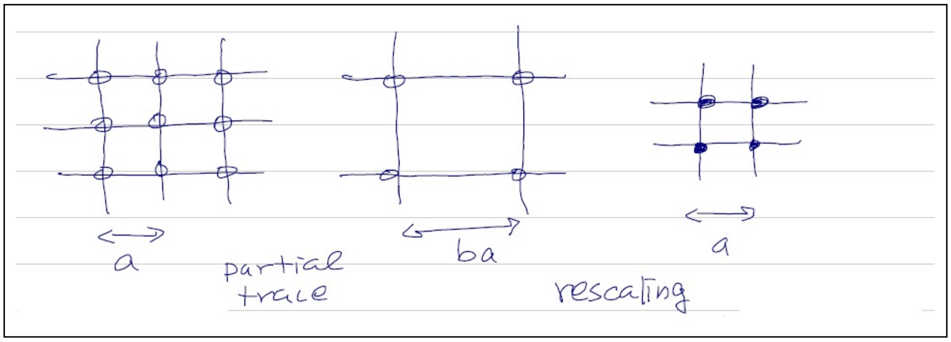
- As for y_t, we have

$$y_t = \dot{R}_1'(t_c) = 2 + \sqrt{2}\log(\sqrt{2}-1) = 0.753549\cdots,$$

slightly closer to $y_t^{\text{exact}} = 1$ than the simple MKRG with $b = 2$.

Better, but still ad-hoc (not obvious how to further improve).

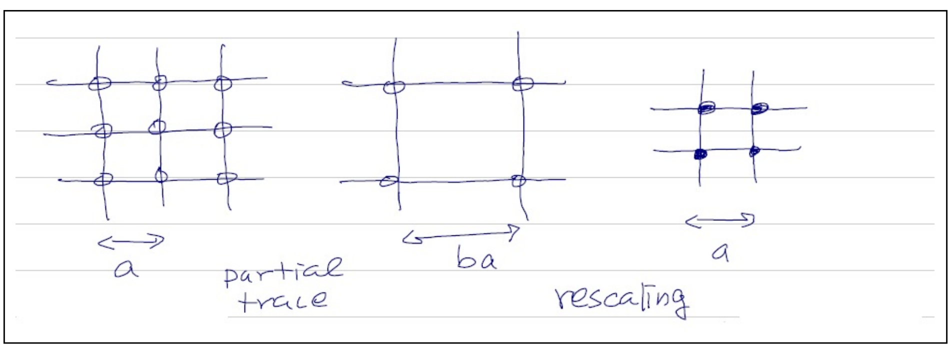
General renormalization group (RG) transformation



- In the derivation of the Ginzburg criterion, we introduced the coarse-graining transformation as a Gedankenexperiment.
- The RG transformation consists of two steps: (i) coarse-graining and (ii) rescaling. Schematically,

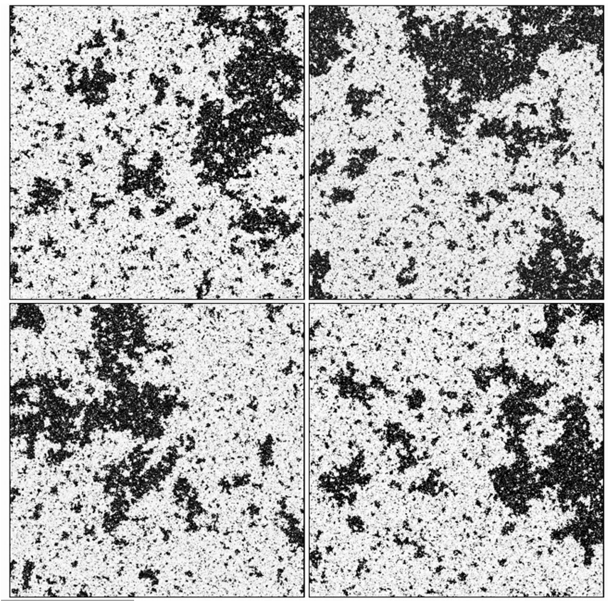
$$\mathcal{H}_a(S \,|\, \boldsymbol{K}, L) \xrightarrow{\text{(i)}} \mathcal{H}_{ab}(\tilde{S} \,|\tilde{\boldsymbol{K}}, L) \xrightarrow{\text{(ii)}} \mathcal{H}_a(S' \,|\boldsymbol{K}', b^{-1}L)$$

General Renormalization Group Transformation



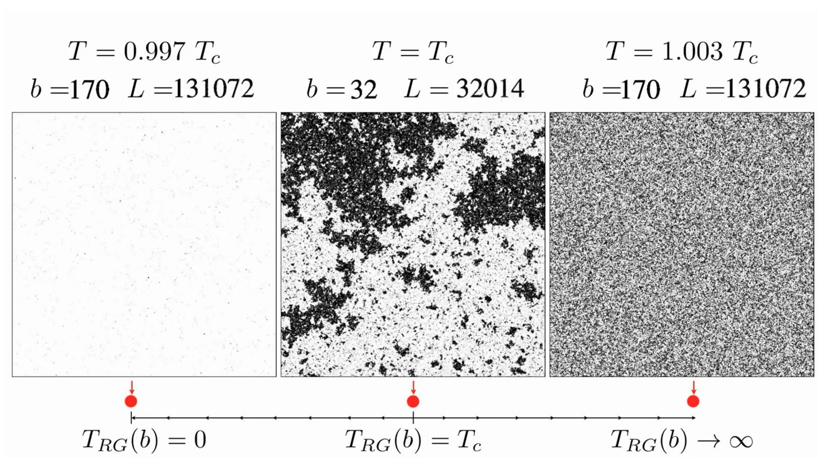
- In the coarse-graining step, we define coarse-grained field and carry out the configuration-space summation of the partition function, with the constraint imposed by the coarse-grained fields.
- In the rescaling step, we redefine the length-scale and the field variables by multiplying them with scaling factors so that the effective Hamiltonian may be the same form as the original one.

Critical point is scale-invariant



"https://youtu.be/fi-g2ET97W8" by Douglas Ashton

Coarse-graining flow



"https://youtu.be/MxRddFrEnPc" by Douglas Ashton

"Critical point" = "Fixed point of RG transformation"

Exercise 6.1: Try the idea of MKRG (i.e., bunching up and trace over intermediate spins) on the Ising model in higher dimensions ($d > 2$).