Lecture 5: Introduction to Renormalization Group

Naoki KAWASHIMA

ISSP, U. Tokyo

May 19, 2025

In this lecture we see ...

- There are cases where we can rely on the mean-field theory even for the critical behavior. (Ginzburg criterion)
- However, in low dimensions including $d = 3$, the mean-field theory is not self-consistent concerning the critical phenomena.
- We can define the renormalization group (RG) transformation, and if we can calculate its result, we would be able to discuss the critical properties of the system.
- For 1D Ising model, we can carry out the RG transformation, which yields correct critical behavior.

When can MF be valid? — Ginzburg criterion

- First, we will elucidate the meaning of the asymptotic validity and draw a general criterion.
- Then, we will check whether the mean-field theory satisfies the criterion in a self-consistent way.
- We will find that it is indeed self-consistent in some cases, but not in general. (Ginzburg criterion)

Typically, MF approximation is bad at microscopic scale

- The mean-field (MF) description should be valid when the relative fluctuation is negligible, i.e., $|\delta\phi_{\boldsymbol{x}}| \ll |\langle\phi_{\boldsymbol{x}}\rangle|$
- Very slightly below the critical temperature, $T = T_{\mathrm{c}} - \delta$, $(0 < \delta \ll 1)$, the spontaneous magnetization is small $0 < |m| \equiv |\langle\phi_{\boldsymbol{x}}\rangle| \ll 1$.
- On the other hand, the fluctuation $|\delta\phi_{\boldsymbol{x}}| \sim |\phi_{\boldsymbol{x}} - m|$ is not typically small. For example, for the Ising model, $\phi_{\boldsymbol{x}} = S_{\boldsymbol{x}} = \pm 1$, which means $|\delta\phi_{\boldsymbol{x}}| \sim |\pm 1 - m| \sim 1$ when $|m| \ll 1$.

However, it may be asymptotically good

- Though the condition is **not** usually satisfied at the scale of lattice constant, a, it may still be correct at some larger length-scales b.
- So, instead of $\phi_{\boldsymbol{x}}$, we consider the average over the cluster of size b, $\phi_{\boldsymbol{X}} \equiv \frac{1}{b^d}\sum_{\boldsymbol{x}\in\Omega_b(\boldsymbol{X})}\phi_{\boldsymbol{x}}$ such that $a \ll b \ll \xi$ ($=$ correlation length).
- Suppose that the MF picture is asymptotically correct. It means that, in some temperature region $T_{\rm c} - \Delta T < T < T_{\rm c}$,
  1. the relative fluctuation $|\delta\phi_{\boldsymbol{X}}|/|\phi_{\boldsymbol{X}}|$ is small, and
  2. the mean-field description is valid for long-range behavior of the system and produces the correct scaling behaviors. Specifically, the order parameter m has the form $m^2 \sim |t-t_c|/u \sim \rho/(u\xi^2)$, and the two-point correlation function at the distance larger than $\lambda\xi$ obeys the Ornstein-Zernike form where λ is some small but finite constant ($0 < \lambda \ll 1$) independent of the temperature or ξ.

Self-consistency of mean-field approximation

Now, let us examine whether these conditions can be met simultaneously.

- For $\langle\phi_{\boldsymbol{X}}\rangle$, below T_c, we have $\langle\phi_{\boldsymbol{X}}\rangle^2_{\rm MF} \sim m^2 \sim \dfrac{|\Delta t|}{u} \sim \dfrac{\rho}{u\xi^2}$
- As for the amplitude of the fluctuation, for $\lambda \ll b/\xi \ll 1$

$$\langle(\delta\phi_{\boldsymbol{X}})^2\rangle_{\rm MF} = \left(\frac{a}{b}\right)^{2d}\sum_{\boldsymbol{x},\boldsymbol{x}'\in\Omega_b(\boldsymbol{X})}\langle\delta\phi_{\boldsymbol{x}'}\delta\phi_{\boldsymbol{x}}\rangle \lesssim \frac{A(b/\xi)}{\rho\xi^{d-2}} \quad (* \text{ see supplement})$$

  where $A(x)$ is dimensionless and finite for any $0 < x < 1$.
- It follows that, when $d < 4$, if we send $\xi \to \infty$ with fixed b/ξ, $\langle\delta\phi^2_{\boldsymbol{X}}\rangle_{\text{MF}}/\langle\phi_{\boldsymbol{X}}\rangle^2_{\text{MF}} \sim A(b/\xi)u\rho^{-2}\xi^{4-d}$ always diverge.

Ginzburg criterion (Upper critical dimension)

At $d < 4$, the MF approximation to the ϕ^4 model **cannot** be asymptotically self-consistent. (On the other hand, the validity of the MF description has been well established by numerical calculation and RG theories for $d > 4$.)

Supplement: MF estimate of fluctuation

From the assumption, the correlation function must obey the OZ form.

$$\langle \delta\phi_{\boldsymbol{x}+\boldsymbol{r}}\delta\phi_{\boldsymbol{x}}\rangle \sim \frac{1}{\rho}\frac{\kappa'^{d-2}}{(\kappa' r)^{\frac{d-1}{2}}}e^{-\kappa'|\boldsymbol{r}|},\ (r > \lambda\xi) \quad \left(\kappa' = \frac{1}{\xi} \approx \sqrt{-\Delta t}\right)$$

from which we obtain for b such that $\lambda\xi \ll b \ll \xi$,

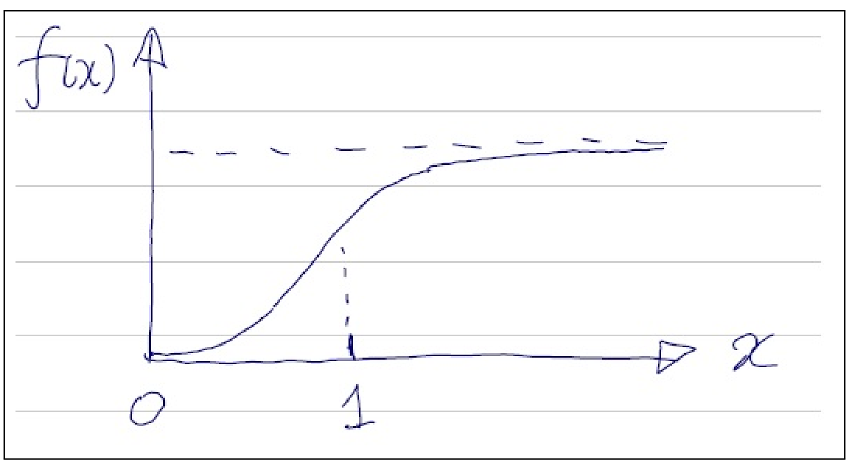
$$\langle(\delta\phi_{\boldsymbol{X}})^2\rangle = \left(\frac{a}{b}\right)^{2d}\sum_{\boldsymbol{x},\boldsymbol{x}'\in\Omega_b(\boldsymbol{X})}\langle\delta\phi_{\boldsymbol{x}'}\delta\phi_{\boldsymbol{x}}\rangle \sim \left(\frac{a}{b}\right)^{d}\sum_{\Delta\boldsymbol{r}}\frac{\rho^{-1}\kappa'^{d-2}}{(\kappa'|\Delta\boldsymbol{r}|)^{\frac{d-1}{2}}}e^{-\kappa'|\Delta\boldsymbol{r}|}$$

$$\sim \frac{1}{b^d}\int_0^b dr\, r^{d-1}\frac{\rho^{-1}\kappa'^{d-2}}{(\kappa' r)^{\frac{d-1}{2}}}e^{-\kappa' r} \sim \frac{1}{b^d}\frac{1}{\rho\kappa'^2}\int_0^{\kappa' b}dx\, x^{\frac{d-1}{2}}e^{-x}$$

$$\sim \frac{f(\kappa' b)}{\rho\kappa'^2 b^d} \quad \left(f(x) \sim \begin{cases} x^{\frac{d+1}{2}} & (x \ll 1) \\ f_\infty \text{ (a dimension-less constant)} & (x \gg 1)\end{cases}\right)$$

$$\sim \frac{\kappa'^{2-d}}{\rho} \times \frac{f(\kappa' b)}{(\kappa' b)^d} = \frac{A(b/\xi)}{\rho\xi^{d-2}}. \quad \left(A(x) \equiv \frac{f(x)}{x^d}\right)$$

Supplement: MF estimate of fluctuation



Coarse-graining

In the coarse-graining step of the RG procedure, we first define "coarse-grained field", $\tilde{S}_{\boldsymbol{R}}$, which is defined in terms of $S_{\boldsymbol{r}}$ in the neighborhood of $\boldsymbol{R}$, i.e., $\tilde{S}_{\boldsymbol{R}} = \Sigma(\{S_{\boldsymbol{r}}\}_{\boldsymbol{r}\in\Omega_b(\boldsymbol{R})})$, with some function $\Sigma(\cdots)$. More formally,

$$e^{-\mathcal{H}_a(S|\boldsymbol{K},L)} \to e^{-\mathcal{H}_{ab}(\tilde{S}|\tilde{\boldsymbol{K}},L)} \equiv \sum_S \Delta(\tilde{S}\,|\Sigma(S))e^{-\mathcal{H}_a(S|\boldsymbol{K},L)},$$

where $\boldsymbol{K}$ is a set of parameters such as $\boldsymbol{K} \equiv (\beta, H)$.

Example 1 (Ising chain with $b = 3$)

$$\Sigma(S_1, S_2, S_3) = S_2 \qquad \text{(Simple decimation)}$$
$$\Sigma(S_1, S_2, S_3) = (S_1 + S_2 + S_3)/3 \qquad \text{(Local Average)}$$
$$\Sigma(S_1, S_2, S_3) = \mathrm{sign}(S_1 + S_2 + S_3) \qquad \text{(Majority rule)}$$

Example: Coarse-graining of Ising chain ($b = 2$)

- Consider the Ising model of size $L \equiv 2^g$ in one dimension.

$$\mathcal{H}_a(S|\boldsymbol{K}, L) = -K\sum_{i=0}^{L-1} S_iS_{i+1} - h\sum_{i=0}^{L-1} S_i \qquad (\boldsymbol{K} \equiv (K, h))$$

- For even L, let us adopt the decimation for the coarse-graining:

$$\tilde{S}_i = S_i \quad (\text{for } i = 0, 2, 4, \cdots, L-2)$$

- Then, $e^{-\mathcal{H}_{2a}(\tilde{S}|\tilde{K},L)} = \sum_{S_1,S_3,\cdots,S_{L-1}} e^{-\mathcal{H}_a(S|K,L)}$. For $h = 0$ we have

$$e^{-\mathcal{H}_{2a}(\tilde{S}|\tilde{K},L)} = \sum_{S_1} e^{K(S_0+S_2)S_1}\sum_{S_3} e^{K(S_2+S_4)S_3}\cdots\sum_{S_{L-1}} e^{K(S_{L-2}+S_0)S_{L-1}}$$
$$\sim e^{\tilde{K}S_0S_2}e^{\tilde{K}S_2S_4}\cdots e^{\tilde{K}S_{L-2}S_0} \sim e^{-\mathcal{H}_{2a}(\tilde{S}|\tilde{K},L)} \qquad (\mathrm{th}\,\tilde{K} \equiv (\mathrm{th}\,K)^2)$$

Example: Rescaling of Ising chain ($b = 2$)

- Let us use $t \equiv \operatorname{th} K$ instead of K for the parameter. Then, the effect of the coarse-graining on t is

$$\tilde{t} = t^2.$$

- The rescaling in the present case is simply

$$\boldsymbol{r}' \equiv \boldsymbol{r}/2, \quad S'_{\boldsymbol{r}'} \equiv \tilde{S}_{\boldsymbol{r}}, \quad \text{and} \quad t' \equiv \tilde{t}.$$

- Together with the coarse-graining, we obtain the whole RG transformation,

$$\mathcal{H}_a(S|t, L) \xrightarrow[b=2]{RG} \mathcal{H}_a(S'|t', L/2), \quad \text{with} \quad t' = t^2.$$

Example: Critical exponent ν

- From the whole RG procedure, we can deduce

$$e^{-r/\xi(t)} \sim \langle S_{\boldsymbol{r}} S_{\boldsymbol{0}} \rangle_t = \langle S_{\boldsymbol{r}'} S_0 \rangle_{t'} \sim e^{-r'/\xi(t')}$$

- Because $r' = r/2$,

$$\xi(t) = 2\xi(t') \quad \left(t' = t^2\right).$$

- Since $t' = t^2$, if we define $g \equiv -\log t$, the correlation length as a function of g would satisfy

$$\xi(g) = 2\xi(2g).$$

- From this, we can obtain $\xi(g)$ upto a constant factor,

$$\xi(g) \sim \frac{1}{g} \qquad \Rightarrow \nu = 1 \quad \text{(Exact!)}$$

Exercise 5.1: By solving the 1D Ising model, compute the correlation function $G(r) \equiv \langle S_r S_0 \rangle$ and the correlation length ξ. Verify $\xi \propto g^{-1}$ where $g \equiv -\log \operatorname{th} K$. (Hint: The correlation function can be expressed as

$$\langle S_r S_0 \rangle = \operatorname{Tr}\left(T^{L-r} \sigma T^r \sigma\right) / \operatorname{Tr}\left(T^L\right)$$

where T is a 2×2 matrix defined as $T_{S',S} \equiv e^{KS'S}$ and σ is another 2×2 matrix defined as $\sigma \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.)

The matrices T and σ can be diagonalized as

$$T = \begin{pmatrix} e^K & e^{-K} \\ e^{-K} & e^K \end{pmatrix} = U \begin{pmatrix} 2\operatorname{ch} K & 0 \\ 0 & 2\operatorname{sh} K \end{pmatrix} U, \quad \sigma = U \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U,$$

where $U \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Therefore, the correlation function, $C(r) \equiv \langle S_r S_0 \rangle$, of a periodic system of length L can be computed as

$$C(r) = \frac{(2\operatorname{ch} K)^{L-r}(2\operatorname{sh} K)^r + (2\operatorname{sh} K)^{L-r}(2\operatorname{ch} K)^r}{(2\operatorname{ch} K)^L + (2\operatorname{sh} K)^L} = \frac{t^r + t^{L-r}}{1 + t^L}$$

with $t \equiv \operatorname{th} K$. Therefore, in the limit $r \ll L$, the correlation function behaves like $C(r) = t^r$. From this, we obtain $e^{-1/\xi} = t$, or $\xi = 1/\log(1/t) = 1/g$.

This is identical to what we obtained from the coarse-graining of the 1D Ising chain.