Lecture 14: Berezinskii-Kosterlitz-Thouless transition

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XY model in two dimensions

- In two dimensions, continuous spin models cannot have magnetically ordered state with spontaneous symmetry breaking. (Mermin-Wagner theorem)
- The XY model, however, has a strange type of phase transition that does not break the symmetry. (BKT transition)
- We can understand this transition by mapping the model into the Coulomb gas model. In this mapping, the spin vortices in the XY model corresponds to charges.
- By a RGT, we obtain Kosteritz's RG flow equation, that predicts special characters of the BKT transition.

Mermin-Wagner theorem

Theorem 1 (Mermin-Wagner(1966))

In two dimensions, if the system has a continuous symmetry (represnted by a compact connected Lie group), it cannot be spontaneously broken at any finite temperature. *[Pfister, Commun. Math. Phys. 79 181 (1981).]*

- Consider the XY model in two dimensions:

$$\mathcal{H} = -K \sum_{(ij)} \boldsymbol{S}_i \cdot \boldsymbol{S}_j = -K \sum_{(ij)} \cos(\theta_i - \theta_j)$$

  where $\boldsymbol{S}_i \equiv (\cos\theta_i, \sin\theta_i)^\mathsf{T}$.

- The XY model has the $U(1)$ symmetry with respect to the transformation $\theta_i \to \theta_i + \alpha$.
- Does the theorem prohibit the phase transition in the XY model?

Berezinskii-Kosterlitz-Thouless transition

- A theoretical proposal of a new type of phase transition without spontaneous symmetry breaking. (Berezinskii (1971), Kosterlitz-Thouless (1973))
- Later the predicted transition was discovered in a thin film experiment of superfluid He4. (Bishop-Reppy (1978))

Vortices

- A typical configuration of spins at low temperature consists of a smooth texture with vortices.
- The smooth texture allows the approximation,

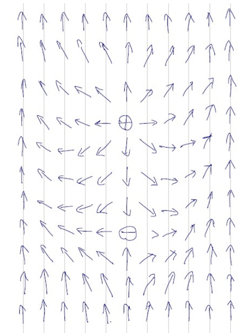
$$\cos(\theta_i - \theta_j) \approx 1 - \frac{1}{2} |\boldsymbol{r}_{ij} \cdot \nabla\theta|^2$$

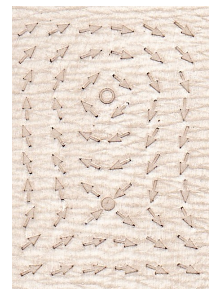
- Therefore, we may switch to continuous space$^{(*)}$

$$\mathcal{H} = -K \sum_{(ij)} \cos(\theta_i - \theta_j)$$
$$\approx \frac{K}{2} \int d\boldsymbol{x} \, |\nabla\theta|^2 + \mu N_{\rm v}$$

where $N_{\rm v}$ is the number of vortices and μN_v comes from the "error" of the continuous approximation that is large near the vortices.

$(*)$ However, we can't forget about the lattice completely as we see later.





Embossed on the souvenir at Prof. Miyashita's retirement party. (June, 2019)

Stationary configuration and fluctuation around it

- Here we introduce a new field variable ϕ that is the deviation of θ from its stationary solution Θ for a given vortex configurations:

$$\theta(\boldsymbol{x}) = \Theta(\boldsymbol{x}) + \phi(\boldsymbol{x}).$$

- The configuration Θ is determined by the condition that $E[\Theta + \delta\Theta] \geq E[\Theta]$ for any function $\delta\Theta(\boldsymbol{x})$:

$$0 \leq E[\Theta + \delta\Theta] - E[\Theta] = \frac{K}{2} \int d\boldsymbol{x} \left\{ |\nabla(\Theta + \delta\Theta)|^2 - |\nabla\Theta|^2 \right\}$$
$$= K \int d\boldsymbol{x} \, \nabla\Theta \cdot \nabla\delta\Theta = -K \int d\boldsymbol{x} \, \triangle\Theta\delta\Theta$$

Therefore, Θ is an harmonic function ($\triangle\Theta = 0$ except at vortices).

- Θ can be uniquely determined (except for the gauge degrees of freedom) by the vortex configuration.

Vortex/fluctuation separation

- Using Θ, we can separate the vortices from the Gaussian fluctuation:

$$\mathcal{H} = \frac{K}{2} \int d\boldsymbol{x} \; |\nabla(\Theta + \phi)|^2 + \mu N_{\rm v} = \mathcal{H}_{\rm v} + \mathcal{H}_{\rm G}.$$

where

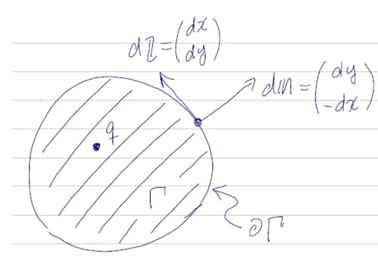
$$\mathcal{H}_{\rm v} \equiv \frac{K}{2} \int d\boldsymbol{x} \; |\nabla\Theta|^2 + \mu N_{\rm v}$$
$$\mathcal{H}_{\rm G} \equiv \frac{K}{2} \int d\boldsymbol{x} \; |\nabla\phi|^2$$

(The ϕ-linear must vanish because of the stationary condition of Θ.)

Vortex field Ψ

- Since Θ is a harmonic function, another harmonic function Ψ must exist such that $\partial\Psi/\partial x = -\partial\Theta/\partial y$, and $\partial\Psi/\partial y = \partial\Theta/\partial x$.
- For a region Γ that includes a vortex,

$$\int_\Gamma d\boldsymbol{x} \; \triangle\Psi = \int_{\partial\Gamma} d\boldsymbol{n} \cdot \nabla\Psi$$
$$= -\int_{\partial\Gamma} d\boldsymbol{l} \cdot \nabla\Theta = -2\pi q$$



where $q = \pm 1, \pm 2, \cdots$ is the vortex charge.

- This (together with $\triangle\Psi = 0$) means

$$\triangle\Psi = -\sum_i 2\pi q_i \delta(\boldsymbol{x} - \boldsymbol{x}_i) = -2\pi\rho_{\rm v}(\boldsymbol{x})$$

Coulomb gas

- Using $G(\boldsymbol{x}) \approx \frac{1}{2\pi}\log\frac{\Lambda}{r}$ that satisfies $\triangle G(\boldsymbol{x}) = -\delta(\boldsymbol{x})$,

$$\Psi(\boldsymbol{x}) = 2\pi \int d\boldsymbol{y}\, G(\boldsymbol{x}-\boldsymbol{y})\rho_{\rm v}(\boldsymbol{y}). \quad \left(\rho_{\rm v}(\boldsymbol{x}) = \sum_i q_i \delta(\boldsymbol{x}-\boldsymbol{x}_i)\right)$$

- The first term in $\mathcal{H}_{\rm v}$ can be reformed as 2D Coulomb Gas:

$$\begin{aligned}
&\frac{K}{2}\int d\boldsymbol{x}\,|\nabla\Theta|^2 = \frac{K}{2}\int d\boldsymbol{x}\,|\nabla\Psi|^2 \\
&\quad = -\frac{K}{2}\int d\boldsymbol{x}\,\Psi\triangle\Psi = \pi K \int d\boldsymbol{x}\,\Psi\rho_{\rm v} \\
&\quad = 2\pi^2 K \int d\boldsymbol{x}d\boldsymbol{y}\, G(\boldsymbol{x}-\boldsymbol{y})\rho_{\rm v}(\boldsymbol{x})\rho_{\rm v}(\boldsymbol{y}) \\
&\quad = 2\pi^2 K \sum_{i,j} G(\boldsymbol{x}_i - \boldsymbol{x}_i) q_i q_j \approx \pi K \sum_{i,j} q_i q_j \log\frac{\Lambda}{|\boldsymbol{x}_i - \boldsymbol{x}_j|} \qquad (1)
\end{aligned}$$

Regularization

- In (1), we have infinities for $i = j$ and singularities for $\boldsymbol{x}_i - \boldsymbol{x}_j \to 0$.
- We must recall that our original problem is a lattice problem.
- The lattice version of Green's function has no infinity at $r = 0$. Therefore, we simply assume $G(0)$ is finite, which allows us to let the $i = j$ terms ($\equiv 2\pi^2 K^2 G(0)\sum_i q_i^2$) absorbed in the μ-term. (We can replace $\sum_i q_i^2$ by N_v because the $|q_i| \geq 2$ terms are energetically and entropically unfavorable and therefore irrelevant.)
- In the lattice model, no two charges can not be close to each other than the lattice constant. Therefore, in (1), we can replace $\sum_{i,j}$ by $2\sum_{(ij)}$ with the condition that the state summation is to be taken over the region where $\boldsymbol{x}_i - \boldsymbol{x}_j > a$ for any i, j pair.

Short summary

The XY model Hamiltonian is approximated by $\mathcal{H}_{XY} \approx \mathcal{H}_{\rm v} + \mathcal{H}_{\rm G}$ where

$$\mathcal{H}_{\rm G} \equiv \frac{K}{2} \int d\boldsymbol{x}\ |\nabla\phi|^2$$

$$\mathcal{H}_{\rm v} \approx \mathcal{H}_{\rm Coulomb} \equiv 2\pi K \sum_{(ij)} q_i q_j \log \frac{\Lambda}{|\boldsymbol{x}_i - \boldsymbol{x}_j|} + \mu N_{\rm v}$$

$$\mathcal{H}_{\rm G} \equiv \frac{K}{2} \int d\boldsymbol{x}\ |\nabla\phi|^2$$

with the constraint on the region of the state summation that no two charges do not come closer than the distance a.

Vortices form two-dimensional Coulomb gas.

Grand partition function (J. M. Kosterlitz: J. Phys. C 7 1046 (1974))

We assume that $q_i = \pm 1$ since vortices $|q_i| > 1$ are energetically and entropically unfavorable and do not contribute. Then, defining

$$g \equiv 2\pi K, \quad \zeta \equiv e^{\mu},$$

the grand partition function becomes

$$\Xi(g,\zeta) = \sum_N \frac{\zeta^{2N}}{(N!)^2} \int_{\Omega(a)} dX_N dY_N\, e^{-gV_N(X_N,Y_N)}$$

$$\Omega(a) \equiv \{\ (X_N, Y_N)\ |\ \text{any two charges are separated by more than } a\}$$

$$V_N(X_N,Y_N) \equiv \sum_{(ij)} v(\boldsymbol{x}_i,\boldsymbol{x}_j) \sum_{(ij)} v(\boldsymbol{y}_i,\boldsymbol{y}_j) - \sum_{ij} v(\boldsymbol{x}_i,\boldsymbol{y}_j)$$

$$v(\boldsymbol{x},\boldsymbol{y}) \equiv \log(\Lambda/|\boldsymbol{x}-\boldsymbol{y}|)$$

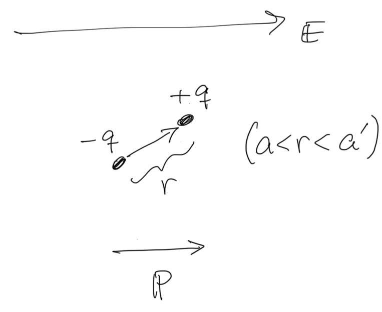
where $X_N \equiv (\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N)$ and $Y_N \equiv (\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_N)$ are the positions of positive and negative vortices, respectively.

Partial trace over charge pairs of size $r < a' = (1+\lambda)a$

- In the lowest order, g is unchanged by RGT, because of the character of the logarithmic potential ($g \log r = g \log r'$ $+ g \log b = g \log r' + g\lambda$ with no change in the prefactor of $\log$).
- The second-order change in g arises from the partial trace over $\Omega(a) \setminus \Omega(a')$. The main contribution is the screening effect by the positive-negative charge pairs with the distance $a < r < a'$. Therefore, it is proportional to the squared charge density:

$$g' \sim g - \lambda A \zeta^2.$$

(A more detailed derivation will follow.)



Screening effect (A simple derivation)

- In the RG from a to $a' = (1+\lambda)a$, pairs of opposite charges, i.e., dipoles with moment qa, are traced out, modifying g.
- We can consider g as $g \equiv \beta/\epsilon$ with the inverse temperature β and the dielectric constant ϵ. With no screening, $g = \beta/\epsilon_0$.
- Generally, when the electric field $\boldsymbol{E}$ induces the polarization density $\boldsymbol{P}$, we get $\boldsymbol{D} \equiv \epsilon \boldsymbol{E} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P}$. Thus, $\Delta\epsilon \equiv \epsilon - \epsilon_0 = |\boldsymbol{P}|/|\boldsymbol{E}|$.
- $\boldsymbol{P} = \rho_d \langle \boldsymbol{d} \rangle$ where ρ_d is the density of the charge pairs and $\boldsymbol{d}$ is the dipole moment of a pair.
- $\rho(\equiv$ charge density$) \propto \zeta \Rightarrow \rho_d \sim \rho^2 \int_{a<|\boldsymbol{x}-\boldsymbol{y}|<a(1+\lambda)} d\boldsymbol{x} d\boldsymbol{y} \sim \lambda a^2 \zeta^2$
- $\langle \boldsymbol{d} \rangle \sim \operatorname*{Tr}_{\boldsymbol{d}} e^{\beta \boldsymbol{E}\cdot\boldsymbol{d}} \boldsymbol{d} / \operatorname*{Tr}_{\boldsymbol{d}} e^{\beta \boldsymbol{E}\cdot\boldsymbol{d}} \sim \beta E d^2 \sim \beta E a^2 q^2$
- Therefore, $\Delta\epsilon = |\boldsymbol{P}|/|\boldsymbol{E}| \sim \lambda \beta q^2 a^4 \zeta^2$.
- Therefore, $g' \sim \beta/(\epsilon_0 + \lambda a^4 q^2 \beta \zeta^2) \sim g - \lambda A \zeta^2$ with $A \sim a^4 q^2 g^2$.

Rescaling

- The first factor contributing to ζ' comes from the Jacobian $d(X_N, Y_N)/d(X'_N, Y'_N) = (a'/a)^{2dN}$, which contributes a factor $e^{d\lambda}$ to ζ'.
- The second factor contributing to ζ' comes from the substitution of (X_N, Y_N) by (bX'_N, bY'_N) in the Coulomb interaction. The Hamiltonian has $N(N-1)$ terms like $-g\log(\boldsymbol{x}_i - \boldsymbol{x}_j)$ and N^2 terms like $g\log(\boldsymbol{x}_i - \boldsymbol{y}_j)$. When $\boldsymbol{x}_i$ is replaced by $b\boldsymbol{x}'_i$, each log produces λ because $\log(\boldsymbol{x}_i - \boldsymbol{x}_j) = \log(\boldsymbol{x}'_i - \boldsymbol{x}'_j) + \lambda$. Therefore, in total, we have the term $g(-N(N-1) + N^2)\lambda = gN\lambda$ in the rescaled Hamiltonian. This amounts to a factor $e^{-g\lambda/2}$ contributing to ζ'.
- Putting these two factors together,

$$\zeta' = \zeta \times e^{(2-g/2)\lambda}$$

RG flow equation

We have obtained the RG flow equations $g' = g - \lambda A\zeta^2$, and $\zeta' = \zeta \times e^{(2-g/2)\lambda}$. It is convenient to use $x \equiv 2 - g/2$ instead of g, and focus on the vicinity of $x = \zeta = 0$.

$$\frac{dx}{d\lambda} = -\frac{1}{2}\frac{dg}{d\lambda} \approx \frac{A}{2}\zeta^2 \quad \text{and} \quad \frac{d\zeta}{d\lambda} = (2-g/2)\zeta = x\zeta$$

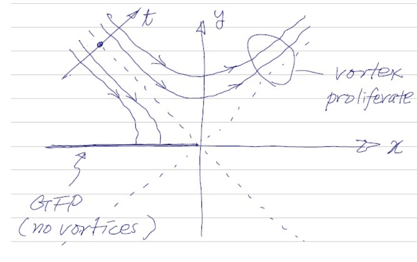
We can remove the factor $A/2$ by defining $y \equiv \sqrt{A/2}\zeta$. Thus we have obtained the famous RG flow equation of the Kosterlitz-Thouless transition:

$$\begin{cases} \dfrac{dx}{d\lambda} = y^2 \\ \dfrac{dy}{d\lambda} = xy \end{cases} \qquad \begin{pmatrix} x = 2 - \pi K \\ y = \text{(const)} \times e^{\mu} \end{pmatrix} \qquad \text{(Kosterlitz's RG eq.)}$$

RG flow diagram

- We can show that $t \equiv y^2 - x^2$ is the constant of motion of the RG equation

$$\frac{dx}{d\lambda} = y^2, \quad \frac{dy}{d\lambda} = xy.$$



- The value of t depends only on the initial values of the parameter, μ and $K = 1/T$. Schematically, the initial points are located on the t axis.
- There are two cases: ($t < 0$) y goes to zero (no vortices) and ($t > 0$) y goes to infinity (vortex proliferation). The separatorix, $t = 0$, corresponds to the BKT transition.

Solution and correlation length

- In the case where $t \equiv y^2 - x^2 > 0$, $\dfrac{dx}{d\lambda} = y^2 = t + x^2$. This equation has the solution $x(\lambda) = \sqrt{t}\,\tan\left(\sqrt{t}\,(\lambda - \lambda_0)\right)$.
- Consider a travel along the RGT flow line with $t > 0$ from $\lambda = 0$ to $\lambda_* \equiv \log(\xi/a)$. ($\xi$ is the correlation length.) Before RGT is applied, we know that no parameter is extreme, which means $x(0), y(0) \sim \mathcal{O}(1)$. At $\lambda = \lambda_*$, the correlation length (= the mean distance between vortices) is a; the vortex density is $\sim a^{-2}$, which leads to $y(\lambda_*) \sim O(1)$.
- When $0 < t \ll 1$, these conditions imply that during this travel the argument of $\tan$ function varies from somewhere close to $-\pi/2$ to somewhere close to $\pi/2$. This means $\pi = \sqrt{t}\lambda_*$.
- This means that

$$\frac{\xi}{a} \sim e^{\frac{\pi}{\sqrt{t}}} \sim \exp\left(\frac{\text{const}}{\sqrt{T - T_c}}\right).$$ (More divergent than any power-law)

Correlation function below the transition temperature

- When $T < T_c$, the system flows to the vortex free states, i.e., it is asymptotically described by the Gaussian fixed-point Hamiltonian.
- Therefore, the 2-point correlation function is

$$\langle S^x(\boldsymbol{x})S^x(\boldsymbol{y}) + S^y(\boldsymbol{x})S^y(\boldsymbol{y})\rangle = \langle e^{i(\phi(\boldsymbol{x})-\phi(\boldsymbol{y}))}\rangle$$
$$= Z_{\mathrm{G}}^{-1}\int d\boldsymbol{\phi}\, e^{-\frac{K}{2}(\nabla\boldsymbol{\phi})\cdot(\nabla\boldsymbol{\phi})-i\boldsymbol{\omega}\cdot\boldsymbol{\phi}} = Z_{\mathrm{G}}^{-1}\int d\boldsymbol{\phi}\, e^{\frac{K}{2}\boldsymbol{\phi}^{\mathsf{T}}\Delta\boldsymbol{\phi}-i\boldsymbol{\omega}\cdot\boldsymbol{\phi}}$$

where $\omega(\boldsymbol{x}) \equiv 1,\ \omega(\boldsymbol{y}) \equiv -1$, and $\omega(\boldsymbol{r}) \equiv 0$ everywhere else. Because of the relation between the lattice Laplacian and the lattice Green's function, $\Delta = -G^{-1}$, and $G(\boldsymbol{x},\boldsymbol{y}) \sim 1/(2\pi)\log(\Lambda/|\boldsymbol{x}-\boldsymbol{y}|)$, the equation can be continued by Gaussian integras as

$$= e^{-\frac{1}{2K}\boldsymbol{\omega}^{\mathsf{T}}G\boldsymbol{\omega}} = e^{-\frac{1}{K}(G(0)-G(r))} \propto r^{-\frac{1}{2\pi K}}.$$

- The correlation decays algebraically not only at the transition point but also any temperature below.

Universal jump

- Thus, we have obtained the correlation function $\sim r^{-\eta}$ with

$$\eta = \frac{1}{2\pi K} = \frac{k_{\mathrm{B}}T}{2\pi J}.$$

This type of correlation is called "quasi-long-range order".
- In particular, at the transition point, $K_c \equiv \frac{2}{\pi}$, the exponent takes a universal value, $\eta(K = K_c) = 1/4$.
- In the context of 2D superfluidity, when it is finite, the superfulid density ρ_{s} is related to K as

$$K = \frac{\hbar^2\rho_{\mathrm{s}}}{mk_{\mathrm{B}}T} \quad \text{or} \quad \rho_{\mathrm{s}} = \frac{mk_{\mathrm{B}}T}{2\pi\hbar^2\eta}$$

where m is the mass of a constituent particle. Therefore, at the BKT transition, ρ_{s} jump from 0 to the universal magnitude $\frac{2mk_{\mathrm{B}}T}{\pi\hbar^2}$.

Summary

- The XY model is mapped to a composite system of vortices and fluctuations.
- The vortices behave as a 2D Coulomb gas.
- The fluctuations are governed by the massless Gaussian model.
- The RGT to the 2D Coulomb gas yields a set of RG flow equation.
- Above the transition temperature, the correlation length diverges as $\xi \sim \exp(c/\sqrt{T - T_c})$.
- Below the transition temperature, the system flows into the vortex-less Gaussian FP, where the spin-spin correlation obeys power-law with the exponent η varying with temperature.
- Its value is $1/4$ at the transition point. This means the universal jump in the superfluid density.

Supplement: The original derivation (Kosterlitz: J.Phys.C7 1046 (1974))

- In what follows, we assume that $q_i = \pm 1$ since vortices $|q_i| > 1$ are energetically unfavorable and would not yield dominant contribution.
- $X_N \equiv (\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N)$ and $Y_N \equiv (\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_N)$ are the positions of positive and negative vortices, respectively.
- Then, the grand partition function is

$$\Xi(g, \zeta) = \sum_N \frac{\zeta^{2N}}{(N!)^2} Z_N^a(g) \quad (\zeta \equiv e^{\mu})$$

$$Z_N^a(g) \equiv \int_{\Omega(a)} dX_N dY_N \, e^{-gV_N(X_N, Y_N)} \quad (g \equiv 2\pi K)$$

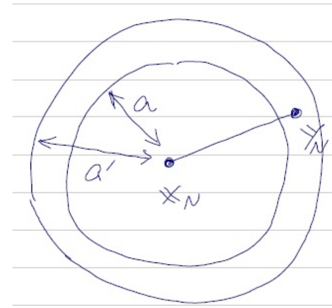
$$\Omega(a) \equiv \{ \ (X_N, Y_N) \mid \text{Any two elements are apart by more than } a \ \}$$

$$V_N(X_N, Y_N) \equiv -\sum_{(ij)} (v(\boldsymbol{x}_i, \boldsymbol{x}_j) + v(\boldsymbol{y}_i, \boldsymbol{y}_j)) + \sum_{ij} v(\boldsymbol{x}_i, \boldsymbol{y}_j)$$

$$v(\boldsymbol{x}, \boldsymbol{y}) \equiv \log(|\boldsymbol{x} - \boldsymbol{y}|/\Lambda)$$

Supplement: Partial trace — Increasing the cut-off a

- Following the general program of the RGT, we first want to take the partial trace with respect to the short-scale degrees of freedom.
- We take the partial integral over the region $\Delta\Omega(a) \equiv \Omega(a) - \Omega(a')$ where $a' \equiv (1+\lambda)a$.
- The region consists of 3 components:



$$\Delta\Omega(a) \approx \sum_{ij} \Omega_{ij}^{+-}(a') + \sum_{(ij)} (\Omega_{ij}^{++}(a') + \Omega_{ij}^{--}(a'))$$

$$\Omega_{ij}^{+-}(a') \equiv \{ \ (X_N, Y_N) \in \Omega(a) \ |$$

All pairs are separated by more than a',

except $a < |\boldsymbol{x}_i - \boldsymbol{y}_j| < a'$. $\}$

$$\Omega_{ij}^{++}(a') \equiv \cdots$$

Supplement: Partial trace — Dipole-mediated interaction

The contribution from Ω^{+-} should be dominant.

$$Z_N^a - Z_N^{a'} \approx \sum_{ij} \int_{\Omega_{ij}^{+-}(a')} dX_N dY_N \, e^{-gV_N} = N^2 \int_{\Omega_{NN}^{+-}(a')} dX_N dY_N \, e^{-gV_N}$$

$$= N^2 \int_{\Omega(a')} dX_{N-1} dY_{N-1} \, e^{-gV_{N-1}} \int_{a<|\boldsymbol{x}_N - \boldsymbol{y}_N|<a'} d\boldsymbol{x}_N d\boldsymbol{y}_N \; e^{-g\sum_i [\Delta v(\boldsymbol{x}_i) - \Delta v(\boldsymbol{y}_i)]}$$

$$(\Delta v(\boldsymbol{x}_i) \equiv v(\boldsymbol{x}_i, \boldsymbol{y}_N) - v(\boldsymbol{x}_i, \boldsymbol{x}_N))$$

$$\approx N^2 \int_{\Omega(a')} dX_{N-1} dY_{N-1} \, e^{-gV_{N-1}}$$

$$\times \int_{a<|\boldsymbol{x}_N - \boldsymbol{y}_N|<a'} d\boldsymbol{x}_N d\boldsymbol{y}_N \left\{ 1 + \frac{g^2}{2} \left(\sum_{i=1}^{N-1} (\Delta v(\boldsymbol{x}_i) - \Delta v(\boldsymbol{y}_i)) \right)^2 \right\}$$

$$\underset{(*)}{\approx} N^2 \int_{\Omega(a')} dX_{N-1} dY_{N-1} \, e^{-gV_{N-1}} \times \left(\cancel{2\pi a^2 \lambda L^2} + 4\pi^2 a^4 \lambda g^2 V_{N-1} \right)$$

(contribution to the regular part is omitted)

Supplement: Partial trace — Screening effect

$$\Xi(g,\zeta) = \sum_N \frac{\zeta^{2N}}{(N!)^2} Z_N^a(g)$$

$$\approx \sum_N \frac{\zeta^{2N}}{(N!)^2} \left(Z_N^{a'}(g) + N^2 \int_{\Omega(a')} dX_{N-1} dY_{N-1}\, e^{-gV_{N-1}} \gamma g^2 \lambda V_{N-1} \right)$$

$$(\gamma \equiv 4\pi^2 a^4;\ (N-1) \to N)$$

$$= \sum_N \frac{\zeta^{2N}}{(N!)^2} \int_{\Omega(a')} dX_N dY_N\, e^{-gV_N} \left(1 + \gamma g^2 \lambda \zeta^2 V_N\right)$$

$$\approx \sum_N \frac{\zeta^{2N}}{(N!)^2} \int_{\Omega(a')} dX_N dY_N\, e^{-(g - \gamma g^2 \lambda \zeta^2) V_N}$$

The 2nd order perturbation screens the Coulomb interaction

Supplement: Rescaling of the interaction

We rescale the length so that a' comes back to a.

$$\boldsymbol{x}'_i = \frac{a}{a'} \boldsymbol{x}_i = e^{-\lambda} \boldsymbol{x}_i$$

By this replacement, the interaction becomes

$$V_N(X_N, Y_N)$$

$$= \sum_{(ij)} \left(\log \frac{\Lambda}{\boldsymbol{x}_i - \boldsymbol{x}_j} + \log \frac{\Lambda}{\boldsymbol{y}_i - \boldsymbol{y}_j} \right) - \sum_{ij} \log \frac{\Lambda}{\boldsymbol{x}_i - \boldsymbol{y}_j}$$

$$= \sum_{(ij)} \left(\log \frac{\Lambda}{(\boldsymbol{x}'_i - \boldsymbol{x}'_j)} + \log \frac{\Lambda}{(\boldsymbol{y}'_i - \boldsymbol{y}'_j)} - 2\lambda \right) - \sum_{ij} \left(\log \frac{\Lambda}{(\boldsymbol{x}'_i - \boldsymbol{y}'_j)} - \lambda \right)$$

$$= V_N(X'_N, Y'_N) + (-N(N-1) + N^2)\lambda = V_N(X'_N, Y'_N) + N\lambda$$

Note that we do not have any rescaling factor in front of $V_N(X'_N, Y'_N)$ in contrast to the regular perturbation. Because of this, we do not have the linear term in the RG flow equation, i.e., $y_\mu = 0$.

Supplement: Rescaling

Now, we can summarize the RGT as

$$\begin{aligned}
&\Xi(g,\zeta)\\
&\quad=\sum_N \frac{\zeta^{2N}}{(N!)^2}e^{2dN\lambda}\int_{\Omega(a)} dX'_N dY'_N\, e^{-(g-\gamma g^2\zeta^2\lambda)V_N(X'_N,Y'_N)}e^{-gN\lambda}\\
&\quad=\sum_N \frac{1}{(N!)^2}\left(\zeta e^{(d-\frac{g}{2})\lambda}\right)^{2N}\int_{\Omega(a)} dX_N dY_N\, e^{-(g-\gamma g^2\zeta^2\lambda)V_N(X_N,Y_N)}\\
&\quad=\Xi(\zeta',g')\times e^{\text{(regular term)}}
\end{aligned}$$

where

$$\zeta'=\zeta e^{\left(d-\frac{g}{2}\right)\lambda} \quad\text{and}\quad g'=g-\gamma g^2\zeta^2\lambda$$

In the form of differential equations,

$$\frac{d\zeta}{d\lambda}=\left(2-\frac{g}{2}\right)\zeta \quad\text{and}\quad \frac{dg}{d\lambda}=-\gamma g^2\zeta^2$$

Supplement: Screeing by dimers

$$I\equiv\int_{a<|\boldsymbol{x}_N-\boldsymbol{y}_N|<a'} d\boldsymbol{x}d\boldsymbol{y}\sum_{ij}(\Delta v(\boldsymbol{x}_i)-\Delta v(\boldsymbol{y}_i))(\Delta v(\boldsymbol{x}_j)-\Delta v(\boldsymbol{y}_j))$$

- We use approximation
  $\Delta v(\boldsymbol{r})\equiv\log(\boldsymbol{r}-\boldsymbol{x}_N)-\log(\boldsymbol{r}-\boldsymbol{y}_N),\approx -\frac{\boldsymbol{x}_i-\boldsymbol{x}_N}{|\boldsymbol{x}_i-\boldsymbol{x}_N|^2}\cdot\boldsymbol{d}.\quad(\boldsymbol{d}\equiv\boldsymbol{y}_N-\boldsymbol{x}_N.)$
- Consider a single term

$$\begin{aligned}
I_{ij}&\equiv\int_{a<|\boldsymbol{x}_N-\boldsymbol{y}_N|<a'} d\boldsymbol{x}_N d\boldsymbol{y}_N\sum_{ij}\Delta v(\boldsymbol{x}_i)\Delta v(\boldsymbol{y}_i)\\
&\approx\int d\boldsymbol{x}_N\int_{a<|\boldsymbol{d}|<a'} d\boldsymbol{d}\left(\frac{\boldsymbol{x}_i-\boldsymbol{x}_N}{|\boldsymbol{x}_i-\boldsymbol{x}_N|^2}\cdot\boldsymbol{d}\right)\left(\frac{\boldsymbol{y}_i-\boldsymbol{x}_N}{|\boldsymbol{y}_i-\boldsymbol{x}_N|^2}\cdot\boldsymbol{d}\right)\\
&=2\pi a^4\lambda\int d\boldsymbol{x}_N\frac{\boldsymbol{x}_i-\boldsymbol{x}_N}{|\boldsymbol{x}_i-\boldsymbol{x}_N|^2}\cdot\frac{\boldsymbol{y}_i-\boldsymbol{x}_N}{|\boldsymbol{y}_i-\boldsymbol{x}_N|^2}
\end{aligned}$$

Supplement: Screeing by dimers (2)

$$I_{ij}(\boldsymbol{x}_i, \boldsymbol{y}_j) \approx 2\pi a^4 \lambda \int d\boldsymbol{x}_N \, \frac{\boldsymbol{x}_i - \boldsymbol{x}_N}{|\boldsymbol{x}_i - \boldsymbol{x}_N|^2} \cdot \frac{\boldsymbol{y}_i - \boldsymbol{x}_N}{|\boldsymbol{y}_i - \boldsymbol{x}_N|^2}$$

$$\underset{(*)}{\approx} 8\pi^2 a^4 \log \frac{L}{|\boldsymbol{x}_i - \boldsymbol{y}_j|}$$

$$I = \sum_{ij} (I_{ij}(\boldsymbol{x}_i, \boldsymbol{x}_j) + I_{ij}(\boldsymbol{y}_i, \boldsymbol{y}_j) - I_{ij}(\boldsymbol{x}_i, \boldsymbol{y}_j) - I_{ij}(\boldsymbol{y}_i, \boldsymbol{x}_j))$$

$$= 8\pi^2 a^4 \lambda \left\{ -\sum_{(ij)} (v(\boldsymbol{x}_i, \boldsymbol{x}_j) + v(\boldsymbol{y}_i, \boldsymbol{y}_j)) + \sum_{ij} v(\boldsymbol{x}_i, \boldsymbol{y}_j) \right\}$$

$$= 8\pi^2 a^4 \lambda \times V_{N-1}(X_{N-1}, Y_{N-1})$$

Supplement: An integral formula

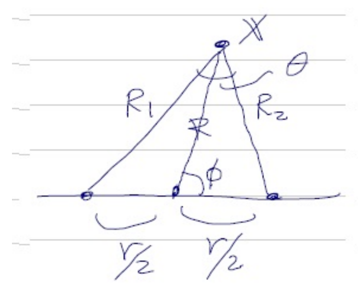
$$I \equiv \int d\boldsymbol{x} \, \frac{\cos\theta}{R_1 R_2}$$

$$= \int d\boldsymbol{x} \, \frac{R^2 - r^2/4}{((\frac{r}{2})^2 + R^2)^2 - r^2 R^2 \cos^2\phi}$$

$$I = \int_0^L dR \, R \, \frac{R^2 - r^2/4}{4}$$

$$\times \int_0^{2\pi} \frac{d\phi}{(r^2/4 + R^2)^2 - r^2 R^2 \cos^2\phi}$$

$$= \int_0^L dR \frac{2\pi R}{R^2 + r^2/4} = \pi \log \frac{L^2 + r^2/4}{r^2/4} \approx 2\pi \log \frac{L}{r}$$



We've used $\int_0^{2\pi} \frac{d\phi}{a + b\cos^2\phi} = \frac{2\pi}{\sqrt{a(a+b)}}$.