Lecture 13: Magnetic Anisotropies

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In this lecture, we see ...

- It is not only $O(n)$ models that we can study by considering the multiple-component field. We can deal with anisotropies as well.

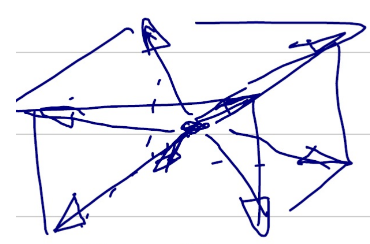
Cubic anisotropy

- Real magnetic systems can never be truely isotropic because spins are coupled with orbital degrees of freedom that are subject to the influence of the lattice.
- In the case of the cubic lattice, for example, the localized spins feel the anisotropy field that has the same symmetry as the cubic lattice.

$$v\left((S_i^x)^4 + (S_i^y)^4 + (S_i^z)^4\right)$$

Decoupled Ising fixed point

- To understand why this term represents the effect of the cubic lattice, consider the case where $v \to \infty$. In this limit, the spin has to point to one of the corners of the unit cell (cube).
- Note that in this limit, the system becomes 3 decoupled Ising models. We will find a fixed point corresponding to this limit.



Scaling operators

- For the ϵ-expansion of the systems with the cubic symmetry, we consider $[\![\cdots]\!]$ of each term in the Hamiltonian.
- t-operator (previously φ_2):

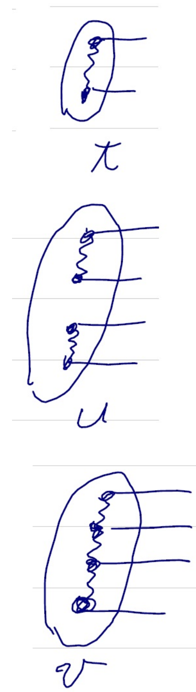
$$\varphi_t \equiv \sum_\alpha [\![\phi_\alpha(\boldsymbol{x})\phi_\alpha(\boldsymbol{x})]\!]$$

- u-operator (previously φ_4):

$$\varphi_u \equiv \sum_{\alpha\beta} [\![\phi_\alpha(\boldsymbol{x})\phi_\alpha(\boldsymbol{x})\phi_\beta(\boldsymbol{x})\phi_\beta(\boldsymbol{x})]\!]$$

- v-operator:

$$\varphi_v \equiv \sum_\alpha [\![\phi_\alpha(\boldsymbol{x})\phi_\alpha(\boldsymbol{x})\phi_\alpha(\boldsymbol{x})\phi_\alpha(\boldsymbol{x})]\!]$$



OPE

- $\varphi_t\varphi_u \approx \cdots + 8\varphi_u + 4(n+2)\varphi_t + \cdots$

$$c_{tu}^t = 4(n+2), \quad c_{tu}^u = 8, \quad c_{tu}^v = 0$$

- $\varphi_t\varphi_v \approx \cdots + 8\varphi_v + 12\varphi_t + \cdots$

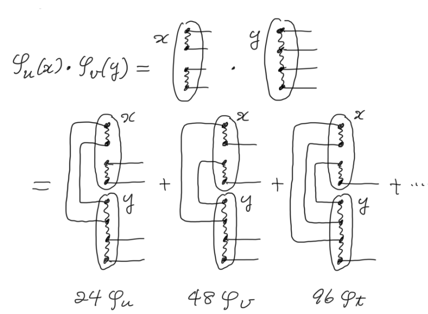
$$c_{tv}^t = 12, \quad c_{tv}^u = 0, \quad c_{tv}^v = 8$$

- $\varphi_u\varphi_v \approx \cdots + 24\varphi_u + 48\varphi_v + 96\varphi_t + \cdots$

$$c_{uv}^t = 96, \quad c_{uv}^u = 24, \quad c_{uv}^v = 48$$

- $\varphi_v\varphi_v \approx \cdots + 72\varphi_v + 96\varphi_t + \cdots$

$$c_{vv}^t = 96, \quad c_{vv}^u = 0, \quad c_{vv}^v = 72$$



RG flow equation

- Keeping in mind that $u = O(\epsilon)$ and $t = O(\epsilon^2)$, as before, the part of the RG flow equation necessary for the lowest order discussion is

$$\begin{cases} \frac{dt}{d\lambda} &= A \equiv \quad 2t - 8(n+2)tu - 24tv + \cdots \\ \frac{du}{d\lambda} &= B \equiv \quad \epsilon u - 8(n+8)u^2 - 48uv + \cdots \\ \frac{dv}{d\lambda} &= C \equiv \quad \epsilon v - 96uv - 72v^2 + \cdots \end{cases}$$

  Note that we have omitted the terms, such as tu in B and u^2 in A, that would not contribute to y_t, y_u, y_v at the non-Gaussian FPs.

- We have four fixed points:
  1. [G] $(t,u,v) = (0,0,0)$
  2. [WF] $(t,u,v) = (t^*_{\mathsf{WF}}, u^*_{\mathsf{WF}}, 0)$
  3. [DI] $(t,u,v) = (t^*_{\mathsf{DI}}, 0, v^*_{\mathsf{DI}})$ ("decoupled Ising FP")
  4. [C] $(t,u,v) = (t^*_{\mathsf{C}}, u^*_{\mathsf{C}}, v^*_{\mathsf{C}})$ ("cubic FP")

Linearization

- In all cases, we have the same form of the linearized RG flow eqs. in terms of $\Delta\boldsymbol{u} \equiv (t - t^*, u - u^*, v - v^*)^{\mathsf{T}}$:

$$\frac{d\Delta\boldsymbol{u}}{d\lambda} = Y\Delta\boldsymbol{u}$$

$$\text{where} \quad Y \equiv \begin{pmatrix} \frac{\partial A}{\partial t} & \frac{\partial A}{\partial u} & \frac{\partial A}{\partial v} \\ \frac{\partial B}{\partial t} & \frac{\partial B}{\partial u} & \frac{\partial B}{\partial v} \\ \frac{\partial C}{\partial t} & \frac{\partial C}{\partial u} & \frac{\partial C}{\partial v} \end{pmatrix}_{t^*,u^*,v^*}$$

$$\equiv \begin{pmatrix} 2 - 8(n+2)u^* - 24v^* & O(\epsilon) & O(\epsilon) \\ O(\epsilon) & \epsilon - 16(n+8)u^* - 48v^* & -48u^* \\ O(\epsilon) & -96v^* & \epsilon - 96u^* - 144v^* \end{pmatrix}$$

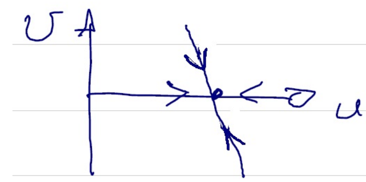
- The lower-right 2×2 sub-matrix is important:

$$\frac{\partial(B,C)}{\partial(u,v)} = \begin{pmatrix} \epsilon - 16(n+8)u^* - 48v^* & -48u^* \\ -96v^* & \epsilon - 96u^* - 144v^* \end{pmatrix}$$

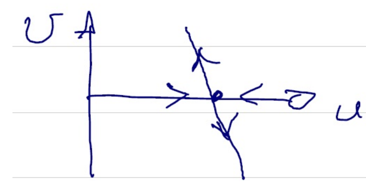
“WF” $\cdots$ $O(n)$ Wilson-Fisher FP

- Within the manifold of $v = 0$, obviously, all results will be the same as before:
$$t^*_{\text{WF}} = \frac{\epsilon^2}{4(n+8)^2} \quad \text{and} \quad u^*_{\text{WF}} = \frac{\epsilon}{8(n+8)} .$$
- The (u, v)-part of the Y matrix becomes
$$\frac{\partial(B,C)}{\partial(u,v)} = \epsilon \times \begin{pmatrix} -1 & -\frac{6}{n+8} \\ 0 & \frac{n-4}{n+8} \end{pmatrix}$$
- The eigenvalues and eigenvectors are
$$y_u^{\text{WF}} \equiv -\epsilon \cdots \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y_v^{\text{WF}} = \frac{n-4}{n+8}\epsilon \cdots \begin{pmatrix} -1 \\ \frac{n+2}{3} \end{pmatrix}$$
- Therefore, we have $n_c \approx 4$ and the WFFP is stable if $n < n_c$.

Case 1: $n < n_c$

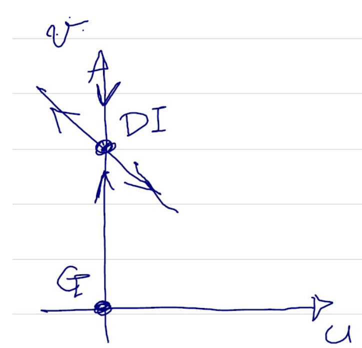


Case 2: $n > n_c$



“DI” $\cdots$ Decoupled Ising fixed point

- Remembering the RG flow equation for v, we find a FP with $u^* = 0$:
$$(u^*_{\text{DI}}, v^*_{\text{DI}}) = \left(0, \frac{\epsilon}{72}\right) .$$
- The (u, v)-part of the Y matrix becomes
$$\frac{\partial(B,C)}{\partial(u,v)} = \begin{pmatrix} \epsilon - 48v^* & 0 \\ -96v^* & \epsilon - 144v^* \end{pmatrix} = \epsilon \cdot \begin{pmatrix} 1/3 & 0 \\ -4/3 & -1 \end{pmatrix}$$
- The eigenvalues and eigenvectors are
$$y_u^{\text{DI}} \equiv \frac{\epsilon}{3} \cdots \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad y_t^{\text{DI}} = -\epsilon \cdots \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



“C” $\cdots$ Cubic fixed point

- Assuming $u, v = O(\epsilon)$ and $t = O(\epsilon^2)$,

$$(u^*_{\mathsf{C}}, v^*_{\mathsf{C}}) = \left(\frac{\epsilon}{24n}, \frac{(n-4)\epsilon}{72n}\right).$$

- The (u, v)-part of the Y matrix becomes

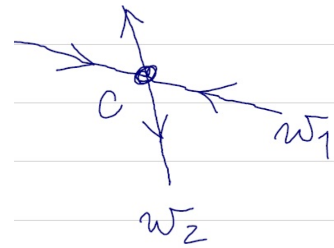
$$\frac{\partial(B, C)}{\partial(u, v)} = -\frac{\epsilon}{3n} \cdot \begin{pmatrix} n+8 & 6 \\ 4(n-4) & 3(n-4) \end{pmatrix}$$

- The eigenvalues and eigenvectors are

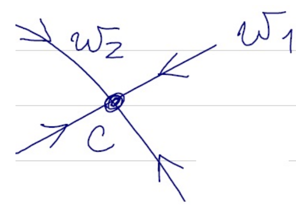
$$y^{\mathsf{C}}_{w_1} = -\epsilon \quad \cdots \begin{pmatrix} 3 \\ n-4 \end{pmatrix},$$

$$y^{\mathsf{C}}_{w_2} = -\frac{n-4}{3n}\epsilon \quad \cdots \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Case 1: $n < n_c$



Case 2: $n > n_c$



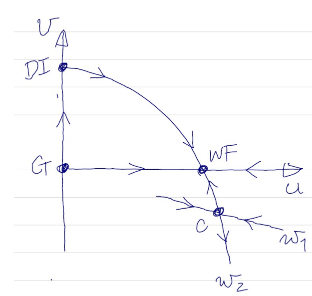
Global structure of RG flow

- Putting together, we can draw the RG flow diagram including the 4 fixed points.

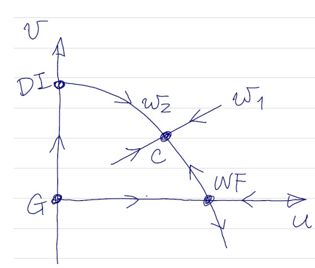
	$n < n_c$ $u^* > 0, v^* < 0$	$n > n_c$ $u^* > 0, v^* > 0$
G	$y_u > 0, y_v > 0$	$y_u > 0, y_v > 0$
WF	$y_u < 0, y_v < 0$	$y_u < 0, y_v > 0$
DI	$y_u > 0, y_v < 0$	$y_u > 0, y_v < 0$
C	$y_{w_1} < 0, y_{w_1} > 0$	$y_{w_2} < 0, y_{w_2} < 0$

- Depending on whether $n < n_c$ or $n > n_c$ we can draw two types of the diagram.
- So, after all the cubic anisotropy is irrelevant for real magnetic systems?

Case 1: $n < n_c$



Case 2: $n > n_c$



Nature of the transition in real magnets

- As we have seen above, the value for n_c is 4 in the lowest order in the ϵ-expansion. According to higher order calculations, it turned out to be close to $n_c \approx 3$ in 3D. So, there has been a long-standing controversy about the nature of the ferromagnetic transition under the cubic anisotropy.
- According to a accurate estimation in [Varnashev: PRB 61 14660 (2000)] $n_c(d=3) < 3$, or more specifically $n_c(d=3) = 2.89(2)$, which suggests that the cubic anisotropy is relevant for the real magnets that are approximately represented by the Heisenberg model.
- For general discussion see [Calabrese et al: arXiv:cond-mat/0509415].

Summary

- By representing the cubic anisotropy by the term $v\sum_\alpha(\phi_i^\alpha)^4$, we have constructed a field theory that may explain the effect of the lattice anisotropy on the spin systems that is otherwise symmetric.
- The ϵ-expansion of the ϕ^4 model with the v term produces a new fixed point. (Cubic fixed point)
- The cubic fixed point is stable for $n > n_c$ whereas it is unstable for $n < n_c$, where $n_c = 4 + O(\epsilon)$.
- According to a more sophisticated numerical estimate, n_c in 3D is slightly below 3, which suggests that we cannot simply neglect the cubic anisotropy in 3D.
- However, the critical region may be narrow in real systems due to smallness of the cubic anisotropy field and the proximity of n_c to 3.

Exercise 12.1: Consider the critical point of the Heisenberg model. Discuss the effect of the uniaxial symmetry breaking-field that is represented by adding the term

$$-D\left[(S_i^z)^2 - \frac{1}{2}((S_i^x)^2 + (S_i^y)^2)\right]$$

to the isotropic Hamiltonian, i.e., the regular Heisenberg model. (Consider the scaling dimension of the scaling operator that corresponds to the above operator, and obtain its scaling dimension at the Wilson-Fisher fixed point for $n = 3$, to the lowest order in ϵ.)