# Lecture 8: More Consequences of Renormalization Group

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## In this lecture, we see ...

- The free energy (and therefore all the quantities derived from it) can be expressed as the sum of a singular part and a regular part.
- The critical phenomena, in particular, the scaling relations among critical exponents, can be systematically derived from the singular part of the free energy.
- By RG flow diagram, we can understand cross-over phenomena.
- From RG, we can derive "finite-size scaling (FSS)," which is useful in predicting how the scaling properties manifest themselves in finite systems.

## Hypothesis: Free energy near the critical point I

In terms of the free energy, the RGT can be expressed as

$$F(\boldsymbol{K}, L) = L^d f(\boldsymbol{K}) + F(\boldsymbol{K}', L/b).$$

If the first term didn't exist, the free energy would be RGT invariant, i.e., $F(\boldsymbol{k}, L) = F(\boldsymbol{k}', L/b)$ . We here postulate that $F(\boldsymbol{K}, L)$ consists of an RGT-invariant term that is responsible for the critical phenomena, and the other term that has nothing to do with it. Formally, we postulate the following:

- The free energy has the following form:

$$F(\boldsymbol{K}, L) = L^d \gamma(\boldsymbol{K}) + F_{\mathrm{s}}(\boldsymbol{K}, L). \tag{1}$$

- The first term, $L^d\gamma$, is purely extensive and non-singular.



## Hypothesis: Free energy near the critical point II

- The second term, $F_{\mathrm{s}}$, is RGT invariant; when $\boldsymbol{K}$ is mapped to $\boldsymbol{K}'$ by a RGT, it satisfies

$$F_{\mathrm{s}}(\boldsymbol{K}', L') = F_{\mathrm{s}}(\boldsymbol{K}, L) \qquad (L' \equiv L/b).$$

- Though $F_{\mathrm{s}}$ is non-singular for any finite $L$, it produces a singular function in the infinite $L$ limit,

$$f_{\mathrm{s}}(\boldsymbol{K}) \equiv \lim_{L\to\infty} L^{-d} F_{\mathrm{s}}(\boldsymbol{K}, L).$$

- The function, $f_{\mathrm{s}}$, represents the critical properties and is singular at the critical point. (For this reason, $F_{\mathrm{s}}$ is called "singular" part of the free energy.)

# Example: 1D Ising model I

- The partition function can be expressed with the transfer matrix as

$$Z = \mathrm{Tr}\, T^L \quad \left( T_{S_1,S_2} \equiv e^{KS_1S_2 + h(S_1+S_2)/2} = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix} \right)$$

- The eigenvalues of $T$ are

$$\lambda_\pm \equiv e^K \left( \mathrm{ch}\, h \pm \sqrt{\mathrm{sh}^2 h + e^{-4K}} \right) \quad (2)$$

- The correlation length is then

$$\xi^{-1} = -\log \frac{\lambda_-}{\lambda_+} \approx 2\sqrt{h^2 + t^2} \qquad (t \equiv e^{-2K}).$$

- $Z = \mathrm{Tr}\, T^L = \lambda_+^L + \lambda_-^L = \lambda_+^L \left( 1 + \left( \frac{\lambda_-}{\lambda_+} \right)^L \right)$



# Example: 1D Ising model II

- $F = Lf + \Delta F \quad \left( f \equiv -\log \lambda_+ ,\ \Delta F \equiv -\log \left( 1 + e^{-L/\xi} \right) \right)$
- Notice that $\Delta F(L/\xi)$ is RGT-invariant because $L'/\xi' = L/\xi$. Therefore, the above equation appear to be the same as (1). However, $f$ (and $\Delta F$ as well) is singular even for finite $L$ because of $\sqrt{\cdots}$ in $\lambda_+$. To make the first and the second term both non-singular, we can exploit the freedom of adding $cL/\xi$ to the first and subtract the same from the second, where $c$ is an arbitrary number. (This change would keep the extensiveness of the first term and RGT invariance of the second, at the same time.)
- Notice also that, for $\xi \gg L$, we have $\Delta F \approx -\log 2 + \dfrac{L}{2\xi}$, which suggest $c = 1/2$.

# Example: 1D Ising model III

- Thus, we have

$$F(\boldsymbol{K}, L) = L\gamma(\boldsymbol{K}) + F_{\mathrm{s}}\left(\frac{L}{\xi}\right)$$

with $\gamma \equiv f + 1/(2\xi)$ and

$$F_{\mathrm{s}} \equiv \Delta F - \frac{L}{2\xi} = -\log\left(2\,\mathrm{ch}\left(\frac{L}{2\xi}\right)\right) \tag{3}$$

In the thermodynamic limit, we obtain the singular function,

$$f_{\mathrm{s}} \equiv \lim_{L\to\infty} \frac{F_{\mathrm{s}}}{L} = -\frac{1}{2\xi} = \frac{1}{2}\log\frac{1-X}{1+X} \quad \left(X \equiv \sqrt{\mathrm{th}^2\, h + \frac{e^{-4K}}{\mathrm{ch}^{\,2} h}}\right)$$



# Finite systems

- The RGT invariance, $F_{\mathrm{s}}(\boldsymbol{K}, L) = F_{\mathrm{s}}(\boldsymbol{K}', L')$ , can be rewritten in terms of scaling field, $u_\mu$,

$$F_{\mathrm{s}}(u_1, u_2, \cdots, L) = F_{\mathrm{s}}(u_1 b^{y_1}, u_2 b^{y_2}, \cdots, L/b). \tag{4}$$

- By setting $b = L/L_0$ where $L_0$ is some constant length scale, and dropping the $L_0$ dependence of the function, we may write

$$F_{\mathrm{s}}(u_1, u_2, \cdots, L) = \tilde{F}_{\mathrm{s}}(u_1 L^{y_1}, u_2 L^{y_2}, \cdots), \tag{5}$$

which is called the "scaling form" of $F_{\mathrm{s}}$.

## Infinite system

- Another way of rewriting $F_\mathrm{s}$ is

$$F_\mathrm{s}(u_1, u_2, \cdots, L) = L^d f_\mathrm{s}(u_1, u_2, \cdots, L) \\ = (L/b)^d f_\mathrm{s}(u_1 b^{y_1}, u_2 b^{y_2}, \cdots, L/b).$$

- Let us assign a special role to the first scaling field, $u_1$, which we assume to be relevant, and denote it as $t$ ($t \equiv u_1, y_t \equiv y_1$).
- By taking $b$ so that $tb^{y_t} = t_0$ is a constant,

$$f_\mathrm{s}(u_1, u_2, \cdots, L) = t^{\frac{d}{y_t}} f_\mathrm{s}(t_0, u_2 t^{-\frac{y_2}{y_t}}, \cdots, L t^{\frac{1}{y_t}})$$

- In the thermodynamic limit, the $L$ dependence on the both side should vanish. Then, by also droping the $t_0$ dependence,

$$f_\mathrm{s}(u_1, u_2, u_3, \cdots) = t^{\frac{d}{y_t}} \tilde{f}_\mathrm{s}\left(u_2 t^{-\frac{y_2}{y_1}}, u_3 t^{-\frac{y_3}{y_1}}, \cdots\right) \quad (6)$$

which is called the scaling form of the free energy density.



## Example: A case with two relevant fields

- We can derive various scaling properties from (5) (or (6)).
- Below, we consider only the vicinity of the critical point which allows us to set all irrelevant fields zero.
- As an example, we consider the case where we have only two relevant fields, $t \equiv u_1$ and $h \equiv u_2$ (like $t \propto T - T_c$ and $h \propto H$ in the Ising model). So, our singular part of the free energy becomes

$$F_\mathrm{s}(t, h, L) = \tilde{F}_\mathrm{s}(tL^{y_t}, hL^{y_h}) \quad (7)$$

## Specific heat exponent $\alpha$

$$F_s(t,h,L) = \tilde{F}_s(tL^{y_t}, hL^{y_h})$$

- For the "specific heat", we have

$$c(t,L) \propto -\frac{1}{L^d}\left(\frac{\partial^2 F_s}{\partial t^2}\right)_{h\to 0} \sim -L^{-d+2y_t}\tilde{F}_s^{(2,0)}(tL^{y_t},0)$$

$$\left(\tilde{F}_s^{(m,n)} \equiv \frac{\partial^m}{\partial t^m}\frac{\partial^n}{\partial h^n}\tilde{F}_s\right)$$

$$\sim L^{2y_t-d}\left(tL^{y_t}\right)^{-\frac{2y_t-d}{y_t}} \times \left(-\left(tL^{y_t}\right)^{\frac{2y_t-d}{y_t}}\tilde{F}_s^{(2,0)}(tL^{y_t},0)\right)$$

$$= t^{-\frac{2y_t-d}{y_t}}\tilde{c}(tL^{y_t}) \qquad \left(\tilde{c}(x) \equiv -x^{\frac{2y_t-d}{y_t}}\tilde{F}_s^{(2,0)}(x,0)\right)$$

- Since $\lim_{L\to\infty} c(t,L)$ is independent of $L$, $c(t,\infty) \propto t^{-\alpha}$ where

$$\alpha = \frac{2y_t - d}{y_t} = 2 - d\nu \quad \left(\nu \equiv \frac{1}{y_t}\right) \tag{8}$$



## Magnetization and susceptibility exponents, $\beta$ and $\gamma$

- For "magnetization", we have

$$m \propto -\frac{1}{L^d}\left(\frac{\partial F_s}{\partial h}\right)_{h\to 0} = -L^{-d+y_h}F_s^{(0,1)}(tL^{y_t},0)$$

$$= t^{\frac{d-y_h}{y_t}}\tilde{m}(tL^{y_t}) \propto t^{\beta}$$

with $$\beta \equiv \frac{d-y_h}{y_t} \tag{9}$$

- For "magnetic susceptibility", we have

$$\chi \propto -\frac{1}{L^d}\left(\frac{\partial^2 F_s}{\partial h^2}\right)_{h\to 0} = -L^{-d+2y_h}F_s^{(0,2)}(tL^{y_t},0)$$

$$= t^{-\frac{2y_h-d}{y_t}}\tilde{\chi}(tL^{y_t}) \propto t^{-\gamma}$$

with $$\gamma \equiv \frac{2y_h-d}{y_t} \tag{10}$$

## Scaling relations

- From (8), (9) and (10),

$$\alpha + 2\beta + \gamma = 2. \qquad \text{(Rushbrooke)} \tag{11}$$

- Similarly, we can also derive that

$$\gamma = \beta(\delta - 1) \qquad \text{(Griffiths)} \tag{12}$$

where $\delta$ is the exponent that characterize the magnetic-field dependence of the magnetization at the critical temperature,

$$m(t = 0, h) \propto h^{1/\delta}$$

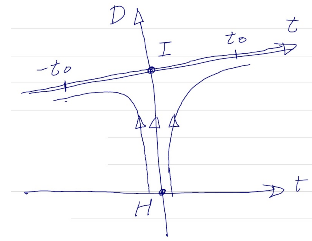


## Crossover phenomena

- A crossover phenomenon is the behavior of the system in which a weak but relevant scaling field manifests itself.
- We can understand it from the scaling form.
- We can also derive the form of the phase boundary near the critical point.

# Flow diagram of crossover phenomena

- We consider two fixed points that are connected by a RG trajectory. Let us label the starting and ending fixed points, "H" and "I", respectively. The scaling field along the trajectory is denoted as $D$. By definition, $D$ is relevant at "H" and irrelevant at "I".




- An example is the 3D classical Heisenberg model

$$\mathcal{H} = -J\sum_{(ij)} \boldsymbol{S}_i \cdot \boldsymbol{S}_j - D\sum_{(ij)} S_i^z S_j^z$$

where $\boldsymbol{S} \equiv (S_i^x, S_i^y, S_i^z)^\mathsf{T}$ ($|\boldsymbol{S}| = 1$). The 3D Heisenberg fixed point "H" is located at $D = 0$, and the 3D Ising fixed point "I" is located at $D = \infty$.



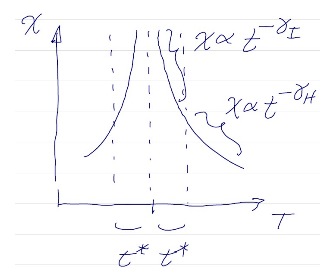
# How crossover affects singularity of quantities

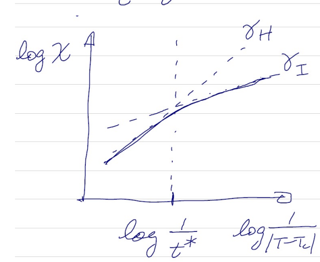
- By (6), the free energy around "H" is

$$f_\mathrm{s}(t, D) = t^{\frac{d}{y_t}} \tilde{f}_\mathrm{s}\left(Dt^{-\phi}\right) \qquad (13)$$

where $\phi$ is crossover exponent $\phi \equiv \dfrac{y_D}{y_t}$.

- Eq.(13) can be re-written as $f_\mathrm{s} = t^{\frac{d}{y_t}} \tilde{f}_\mathrm{s}(t/t^*(D))$ with a "crossover temperature" $t^*(D) \propto D^{1/\phi}$. Then, $f_\mathrm{s}$ behaves like an isotropic Heisenberg model $(D = 0)$ when $t \gg t^*$, whereas it qualitatively deviates from the Heisenberg-like behavior when $t \ll t^*$.



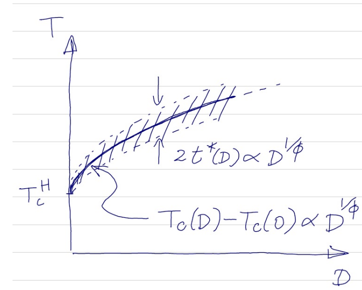

$\gamma_{3\mathrm{DI}} = 1.237075(10)$
$\gamma_{3\mathrm{DH}} \approx 1.35(*)$
$\gamma_{3\mathrm{DXY}} = 1.3177(5)$

$(*)$ Kaupuzs, cond-mat/0101156

## How crossover affects the phase boundary

- Now, we consider the shape of the phase boundary in the $D - T$ phase diagram.
- We again use $f_{\rm s}(t, D) = t^{d/y_t^H} \tilde{f}_{\rm s}(Dt^{-\phi})$ , where $\phi \equiv y_D^H / y_t^H$. where we put superscript "H" to make it clear that $y_t^H$ is the value at H.
- When $D > 0$, the system should show the Ising-like critical behavior. Then, we obtain

$$f_{\rm s}(t, D) \sim (t - t_c(D))^{d/y_t^I}.$$




$\phi \approx 1.2$ for 3D Heisenberg model.

- Now, to satisfy both of these forms at the same time, $f_{\rm s}$ must have the following form near the criticality.

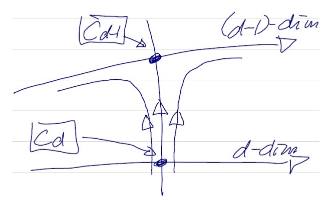
$$f_{\rm s} \propto t^{\frac{d}{y_t^H}} \left(tD^{-\frac{1}{\phi}} - x_0\right)^{\frac{d}{y_t^I}} \propto D^{\frac{d}{\phi}\left(\frac{1}{y_t^H} - \frac{1}{y_t^I}\right)} \left(t - x_0 D^{\frac{1}{\phi}}\right)^{\frac{d}{y_t^I}}$$

Therefore, $t_c(D) \propto D^{\frac{1}{\phi}} = D^{y_t^H / y_D^H}$

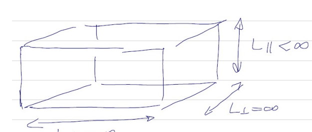


## Dimensional crossover

- Some systems have phase transition even when the size is finite in one direction. However, the critical properties are different from the case where the system is infinite in all directions.

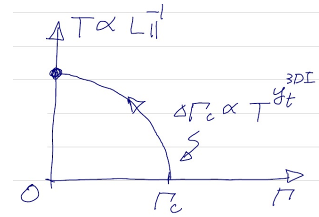



- Though the system size is not a "field" in the conventional terminology, we can treat $D \equiv 1/L_{\|}$, where $L_{\|}$ is the number of layers in the finite direction, as if it were a relevant field.




- Obviously, $y_D$ is 1, because by the RGT, $D \to D' = 1/(L'_{\|}) = 1/(L_{\|}/b) = bD$.
- Therefore, we have $\phi = 1/y_t$ for the crossover exponent, which leads to $t^* \propto L_{\|}^{-y_t}$ for the crossover temperature, and $t_c(L_{\|}) \propto L_{\|}^{-y_t}$ for the transition temperature near the $L_{\|} = \infty$ critical point.

# Quantum critical point




- By Feynman's path integral formulation, $d$-dimensional quantum system can be represented as $(d+1)$-dimensional classical system with size $1/T$ in the new direction.
- In some special cases, the extra dimension, called the "imaginary time", is essentially equivalent to one of the spatial directions.
- For example, the 2-dimensional transverse field Ising model $\mathcal{H} = -J \sum_{(ij)} S_i^z S_i^z - \Gamma \sum_i S_i^x$ has a quantum phase transition at $T = 0$ and $\Gamma = \Gamma_c$, and it can be mapped to 3-dimensional classical Ising model with the size $1/T$ in the 3rd dimension.
- Then, we can apply the dimensional cross-over to this system:

$$t^*(T) \propto t_c(T) \propto L_{\parallel}^{-y_t^{3DI}} \propto T^{y_t} \tag{14}$$

where $t \equiv \Gamma - \Gamma_c$ and $y_t$ is for the 3D Ising model.



# Finite-size scaling

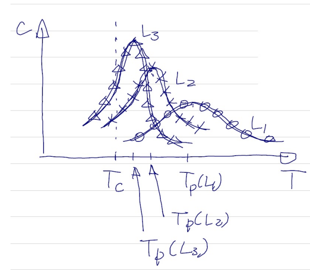
- As we have seen, the RGT-invariant quantity can be used to characterize a critical phenomena.
- We'll see a practical way for obtaining the critical indices (scaling dimensions).
- We can define such a computable RGT-invariant quantity as a difference in the free energy of two system-sizes.

# Specific heat (1)

- Suppose we have obtained $F_{\mathrm{s}}(\boldsymbol{K}, L)$ as a function of $\boldsymbol{K}$ and $L$, and it has the form $F_{\mathrm{s}}(\boldsymbol{K}, L) = \tilde{F}_{\mathrm{s}}(tL^{y_t}, hL^{y_h}, \cdots)$.
- From $\tilde{F}_{\mathrm{s}}$, the scaling form of the specific heat is

$$c \approx \frac{-T}{L^d} \frac{\partial^2 F_{\mathrm{s}}}{\partial T^2} \sim L^{2y_t - d} \tilde{c}(tL^{y_t}). \tag{15}$$

- If $c(T, L)$ diverges at the critical point for $L \to \infty$, we expect that $c$ has a peak around $T \approx T_{\mathrm{c}}$ even if $L$ is finite.
- This is compatible with (15) only if $\tilde{c}$ has a peak itself.






# Specific heat (2)

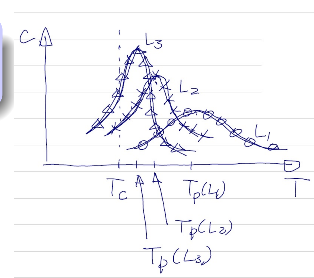
$$c(T, L) \sim L^{2y_t - d} \tilde{c}(tL^{y_t}).$$

- Suppose $\tilde{c}(x)$ has a peak at $x = x_p$. It means that $c(T, L)$ has a peak when $tL^{y_t} = x_p$.
- Let $T_{\mathrm{c}}(L)$ be the temperature at which $c(T, L)$ has the peak. Then,

$$T_{\mathrm{c}}(L) - T_{\mathrm{c}} \propto t_{\mathrm{c}}(L) \propto L^{-y_t}. \tag{16}$$

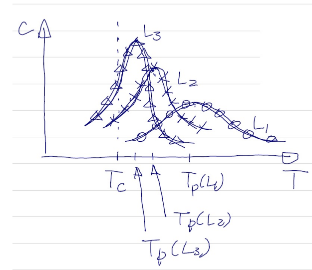
- The height of the peak also carries some information on the critical behavior, i.e., it is proportional to

$$c(T_{\mathrm{c}}(L), L) \propto L^{2y_t - d} \tag{17}$$



## Specific heat (3)

- More directly, by plotting $c/L^{2y_t-d}$ against $(T-T_c)L^{y_t}$, we expect that curves corresponding to varying system sizes fall on top of each other.
- Of course, to do this, we need to choose the right values for $T_c$ and $y_t$, which we do not know initially.
- We can fix these values by some trials-and-errors, like using an analog camera and adjusting the focus.

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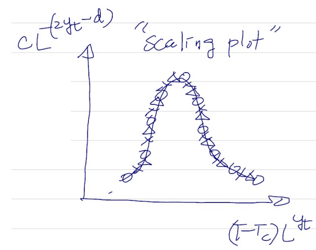




## Remark: Practical substitute of $F_s$

- So far, we have been implicitly assuming that we can compute $F_s$.
- But it is not usually true even for finite systems. That's why people often simply use $F$ itself, more precisely the derivatives of $F$, in the place of those of $F_s$ in numerical calculations. For example, the specific heat itself may be used instead of its singular part.
- However, this "approximation" is bad when the quantity in question is only weakly divergent or even non-divergent, e.g., the specific-heat exponent $\alpha$ is negative in some cases.
- Using (1), we can define a computable RGT-invariant quantity:

$$\begin{aligned}\Delta_b F(\boldsymbol{K}, L) &\equiv (b^d F(\boldsymbol{K}, L/b) - F(\boldsymbol{K}, L))/(b^d - 1) \\ &= (b^d F_s(\boldsymbol{K}, L/b) - F_s(\boldsymbol{K}, L))/(b^d - 1).\end{aligned}$$

  We can use this quantity and its derivatives as the substitute of $F_s$ and its derivatives.

**Exercise 8.1:** Show that the "singular" part of 1D Ising model (3) is non-singular for any finite $L$. (Hint: The singularity comes from the square root in (2). Therefore, if it is squared, it doesn't yield singularity.)



**Exercise 8.2:** Consider the 1-dimensional $S = 1$-Ising model. (The Hamiltonian with the same form as the regular Ising model, but with tri-variate spins, $S_i = -1, 0, 1.$) Following the similar argument as in the lecture, obtain the singular part of the free energy. Confirm that $\xi f_s = -1/2$ for this model, the same as the $S = 1/2$ Ising model.

**Exercise 8.3:** Derive Griffiths' scaling relation (12).

$$m(h) = -\left.\frac{\partial f_s}{\partial h}\right|_{t=0} = L^{y_h - d}\tilde{f}_s(hL^{y_h}) \propto h^{(d-y_h)/y_h}\hat{f}_s(hL^{y_h})$$
$$\Rightarrow \delta = \frac{y_h}{d - y_h}$$

Therefore,

$$\beta(\delta - 1) = \frac{d - y_h}{y_t}\left(\frac{y_h}{d - y_h} - 1\right) = \frac{2y_h - d}{y_t} = \gamma$$