Lecture 7: General Framework of Renormalization Group — Fixed Points and Scaling Operators

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May 31, 2024

In this lecture, we see ...

- Having seen a few examples of the simple real-space RG transformations, in this lecture we formulate it as a general framework for discussing the phase diagram and the critical phenomena.
- As a results, we see that the critical phenomena is characterized by the fixed point of the RGT. In particular, the eigenvalues of the linearized transformation around the fixed point.
- The critical exponents, such as η, ν, β, can be expressed as some simple combinations of scaling eigenvalues.
- As an exactly-treatable example of the RG framework, we consider the Gaussian model, which is easy to solve and provides us the starting point for perturbative renormalization group.

Fixed-point and scaling operators

- We consider a generic Hamiltonian $\mathcal{H}(S|\boldsymbol{K})$ with (generally) many parameters $\boldsymbol{K} = (K_1, K_2, \cdots)$.
- Suppose we have its exact renormalization group transformation (RGT), represented by the change in the parameters, i.e., $\boldsymbol{K} \to \boldsymbol{K'} \equiv R_b(\boldsymbol{K})$. (Even if we cannot actually compute it, we can still make some statements.)
- The function $R_b(\boldsymbol{K})$ defines a "RG flow" in the parameter space, i.e., the set of trajectories in $\boldsymbol{K}$ space along which $\boldsymbol{K}$ moves as we repeatedly apply the RGT. (Roughly speaking, this corresponds to how the appearance of the system changes as it moves farther away from the observer.)
- This RG flow provides us with a framework of understanding the phase diagram.

Generic Hamiltonian

- Any Hamiltonian is expressed as an expansion w.r.t. local operators.

$$\mathcal{H}_a(S|\boldsymbol{K}, L) = -\sum_{\boldsymbol{x}} \sum_{\alpha} K_\alpha S_\alpha(\boldsymbol{x}) \tag{1}$$

where $\{S_\alpha\}$ spans the space of all local operators, i.e.,

$$\forall Q(\boldsymbol{x})\ \exists q_\alpha \ \left(Q(\boldsymbol{x}) = \sum_\alpha q_\alpha S_\alpha(\boldsymbol{x}) \right) \tag{2}$$

- Example: A generic model defined with Ising spins.

$$\begin{array}{ll} K_1 = H & S_1(\boldsymbol{x}) = S_{\boldsymbol{x}} \\ K_2 = J_x & S_2(\boldsymbol{x}) = S_{\boldsymbol{x}}\, S_{\boldsymbol{x}+\boldsymbol{a}_x} \\ K_3 = J_y & S_3(\boldsymbol{x}) = S_{\boldsymbol{x}}\, S_{\boldsymbol{x}+\boldsymbol{a}_y} \\ K_4 = Q & S_4(\boldsymbol{x}) = S_{\boldsymbol{x}}\, S_{\boldsymbol{x}+\boldsymbol{a}_x}\, S_{\boldsymbol{x}+2\boldsymbol{a}_x} \\ K_5 = Q & S_5(\boldsymbol{x}) = S_{\boldsymbol{x}}\, S_{\boldsymbol{x}+\boldsymbol{a}_x}\, S_{\boldsymbol{x}+\boldsymbol{a}_y} \\ \vdots & \vdots \end{array}$$

($\boldsymbol{a}_x, \boldsymbol{a}_y, \cdots$: lattice unit vectors)

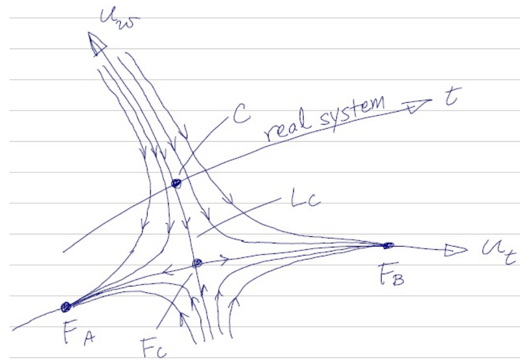
RG flow diagram

- The RGT

$$\mathcal{H}_a(\boldsymbol{S}, \boldsymbol{K}) \to \mathcal{H}_a(\boldsymbol{S}', \boldsymbol{K}')$$

can be regarded as a map from the parameter space onto itself

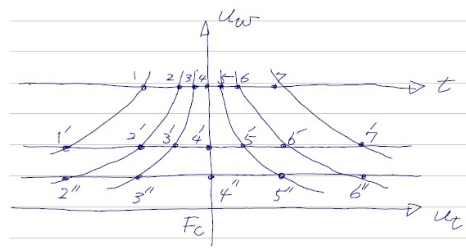
$$\boldsymbol{K} \to \boldsymbol{K}' \equiv R_b(\boldsymbol{K})$$



- An RG trajectory is a RGT-invariant curve.
- We assume that the trajectory is continuous. (In particular, the RGT is defined for continuous b, such that $\mathcal{R}_{b_1}\mathcal{R}_{b_2} = \mathcal{R}_{b_1 b_2}$.)
- A trajectry converging to the unstable fixed point ($\mathrm{F_C}$) is called a critical line ($\mathrm{L_C}$). The parameter along it is called irrelevant (u_w).
- The parameter along a trajectry emanating from the unstable fixed point is called relevant. (u_t).

Critical properties are controlled by unstable fixed-point

We live in the "real world" (the t-axis) with non-zero irrelevant field (u_w), whereas the fixed point (F_c) lies in the "ideal world" (the u_t-axis) with no irrelevant field ($u_w = 0$).



- RGT with b maps the points $1, 2, \cdots, 6, 7$ in the "real world" to $1', 2', \cdots, 6', 7'$, a little closer to the "ideal world" line.
- RGT with $b' > b$ maps points in the narrower region $(2, \cdots, 6)$ to the same range of u_t, but closer to the ideal world.
- The narrower region in the real world is mapped closer the the fixed point, meaning that the asymptotic (i.e., critical) behavior in the real world is governed by the fixed point.
- The amplitude of the irrelevant fields in the real world determines how large-scale we have to go to observe the correct asymptotic behavior.

Expansion around unstable fixed point

- Consider the "local" Hamiltonian at $\boldsymbol{x}$, $H_a(\boldsymbol{S}(\boldsymbol{x})|\boldsymbol{K},\boldsymbol{x})$, with $\boldsymbol{S}(\boldsymbol{x})$ being the subset of $\boldsymbol{S}$ near $\boldsymbol{x}$. Its fixed point form is

$$H_a^*(\boldsymbol{S}(\boldsymbol{x}),\boldsymbol{x}) \equiv H_a(\boldsymbol{S}(\boldsymbol{x})|\boldsymbol{K}^*,\boldsymbol{x}). \quad (3)$$

(In what follows, we focus on the local Hamiltonian, dropping some or all of the parameters, a,$\boldsymbol{x}$ and $\boldsymbol{S}(\boldsymbol{x})$, and use the abbreviation like H^* for $H_a^*(\boldsymbol{S}(\boldsymbol{x})|\boldsymbol{K},\boldsymbol{x})$.)

- Let us denote the RGT symbolically by $\mathcal{R}_b$ where b is the renormalization factor. Then, $\mathcal{R}_b(H^*) = H^*$.
- Let us expand the local Hamiltonian around this fixed point.

$$H = H^* - \sum_\alpha h_\alpha S_\alpha = H^* - \boldsymbol{h}\cdot\boldsymbol{S} \quad (4)$$

where $h_\alpha \equiv K_\alpha - K_\alpha^*$

Linearization of RGT

- Now, consider the transformation applied to the local Hamiltonian near the fixed point:

$$\mathcal{R}_b(H^* - \boldsymbol{h}\cdot\boldsymbol{S}) = H^* - \boldsymbol{h}'\cdot\boldsymbol{S}'$$

- To the lowest order, $\boldsymbol{h}'$ depends linearly on $\boldsymbol{h}$, i.e., a linear operator T_b exists such that

$$\boldsymbol{h}' \approx T_b\boldsymbol{h}.$$

- We assume that T_b is diagonalizable with real eigenvalues.$^{(*)}$

$$P^{-1}T_bP = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix} \equiv \Lambda_b$$

$^{(*)}$ Here we should remember that the initial physical variables, $\boldsymbol{S}$, and the final, $\boldsymbol{S}'$, are different. So, in some cases, the phases of the eigenvalues may be gauged away by redefining $\boldsymbol{S}'$, i.e., if T_b is "rotating", we can rotate $\boldsymbol{S}'$, while $\boldsymbol{S}$ is fixed, so that T_b's eigenvalues are real.

Scaling fields and scaling operators

- By defining

$$\boldsymbol{u} \equiv P^{-1}\boldsymbol{h}, \text{ and } \boldsymbol{\phi} \equiv P^{\mathsf{T}}\boldsymbol{S}$$

we obtain

$$\boldsymbol{u}\cdot\boldsymbol{\phi} = (P^{-1}\boldsymbol{h})^{\mathsf{T}}(P^{\mathsf{T}}\boldsymbol{S}) = \boldsymbol{h}^{\mathsf{T}}(P^{-1})^{\mathsf{T}}P^{\mathsf{T}}\boldsymbol{S} = \boldsymbol{h}\cdot\boldsymbol{S}.$$

In addition, $\boldsymbol{u}$ transforms as

$$\boldsymbol{u}' = P^{-1}\boldsymbol{h}' = P^{-1}T_b\boldsymbol{h} = P^{-1}T_bP\boldsymbol{u} = \Lambda_b\boldsymbol{u},$$

namely, $u'_\mu = b^{y_\mu}u_\mu$ with $y_\mu \equiv \log_b \lambda_\mu$.

$u_\mu =$ "scaling field", $\phi_\mu =$ "scaling operator",

$y_\mu =$ "scaling eigenvalue" $\left(\begin{array}{ll} y_\mu > 0 & \rightarrow u_\mu \text{ is relevant} \\ y_\mu < 0 & \rightarrow u_\mu \text{ is irrelevant} \end{array}\right)$

Scaling dimensions

- In terms of the scaling fields and operators, the local Hamiltonian $H(\boldsymbol{\phi},\boldsymbol{u}) = H^*(\boldsymbol{\phi}) - \boldsymbol{u}\cdot\boldsymbol{\phi}$ is mapped by RGT to

$$H'(\boldsymbol{\phi}',\boldsymbol{u}') = H^*(\boldsymbol{\phi}') - \sum_\mu u'_\mu\phi'_\mu \quad (u'_\mu = b^{y_\mu}u_\mu).$$

- The scaling property of ϕ_μ is determined by y_μ through the condition

$$\sum_{\boldsymbol{x}\in\Omega_b(b\boldsymbol{x}')} u_\mu(\boldsymbol{x})\phi_\mu(\boldsymbol{x}) \approx u'_\mu(\boldsymbol{x}')\phi'_\mu(\boldsymbol{x}') \tag{5}$$

with the box $\Omega_b(b\boldsymbol{x}')$ of size b at $b\boldsymbol{x}'$. This yields

$$\phi'_\mu(\boldsymbol{x}') \approx b^{x_\mu}\phi_\mu(\boldsymbol{x}) \quad \text{with} \quad x_\mu = d - y_\mu \quad \text{(See supplement)}$$

$x_\mu \equiv d - y_\mu =$ "scaling dimension" of ϕ_μ.

Supplement: Meaning of $\phi' = b^x \phi$ with $x = d - y$ I

The equation $\phi' = b^{d-y}\phi$ is symbolic, and may require some more explanation. (Here, the subscript μ is dropped for simplicity.) Precisely, what we mean by this equation is

$$\langle \phi(\boldsymbol{x}_1)\phi(\boldsymbol{x}_2)\rangle_{\mathcal{H}(\phi,u)} \approx b^{-2(d-y)} \left\langle \phi'\left(\frac{\boldsymbol{x}_1}{b}\right)\phi'\left(\frac{\boldsymbol{x}_2}{b}\right)\right\rangle_{\mathcal{H}(\phi',u')}.$$

This justifies the symbolic expression $\phi'(\boldsymbol{x}') = b^x\phi(\boldsymbol{x})$ with $x \equiv d - y$. Using the free energy before and after RGT, i.e., $F \equiv -\log \sum_{\boldsymbol{\phi}} e^{-\mathcal{H}(\boldsymbol{\phi},\boldsymbol{u})}$ and $F' \equiv -\log \sum_{\boldsymbol{\phi}'} e^{-\mathcal{H}(\boldsymbol{\phi}',\boldsymbol{u}')}$, it can be shown as below:

$$\begin{aligned}\langle \phi(\boldsymbol{x}_1)\phi(\boldsymbol{x}_2)\rangle_{\mathcal{H}(\phi,u)} &= \frac{\partial^2 F}{\partial u(\boldsymbol{x}_1)\partial u(\boldsymbol{x}_2)} \\ &\underset{(1)}{\approx} \sum_{\boldsymbol{x}_1',\boldsymbol{x}_2'} \frac{\partial u'(\boldsymbol{x}_1')}{\partial u(\boldsymbol{x}_1)}\frac{\partial u'(\boldsymbol{x}_1')}{\partial u(\boldsymbol{x}_1)}\frac{\partial^2 F'}{\partial u'(\boldsymbol{x}_1')\partial u'(\boldsymbol{x}_2')}\end{aligned}$$

Supplement: Meaning of $\phi' = b^x \phi$ with $x = d - y$ II

$$\begin{aligned}&\underset{(2)}{\approx} \left(\sum_{\boldsymbol{x}'} \frac{\partial u'(\boldsymbol{x}')}{\partial u(\boldsymbol{x})}\right)^2 \times \frac{\partial^2 F'}{\partial u'(\boldsymbol{x}_1/b)\partial u'(\boldsymbol{x}_2/b)} \\ &\underset{(3)}{\approx} b^{-2(d-y)} \left\langle \phi'\left(\frac{\boldsymbol{x}_1}{b}\right)\phi'\left(\frac{\boldsymbol{x}_2}{b}\right)\right\rangle_{\mathcal{H}(\phi',u')}\end{aligned}$$

- The equality (1) holds because F and F' are essentially equal concerning the long-range behaviors.
- The equality (2) holds because of the local nature of RGT, i.e., $u'(\boldsymbol{x}')$ depends only on $u(\boldsymbol{x})$ such that $\boldsymbol{x} \approx b\boldsymbol{x}'$.
- For (3), we must notice that the equation, $u' = b^y u$, actually means that $u'(\boldsymbol{x}') = b^y \bar{u}(b\boldsymbol{x}')$, where $\bar{u}(\boldsymbol{x})$ is the local average of u near $\boldsymbol{x}$. As we see below, $\sum_{\boldsymbol{x}'} \frac{\partial \bar{u}(b\boldsymbol{x}')}{\partial u(\boldsymbol{x})} \approx b^{-d}$, which completes the proof.

Supplement: Meaning of $\phi' = b^x\phi$ with $x = d - y$ III

- Let us define $\chi(\boldsymbol{x}', \boldsymbol{x}) \equiv \frac{\partial \bar{u}(b\boldsymbol{x}')}{\partial u(\boldsymbol{x})}$, and show $A \equiv \sum_{\boldsymbol{x}'} \chi(\boldsymbol{x}', \boldsymbol{x}) = b^{-d}$. (The uniformity of the system and the RGT demands that A is independent of $\boldsymbol{x}$.) Because we are working in the linear approximation, $\bar{u}(b\boldsymbol{x}') = \sum_{\boldsymbol{x}} \chi(\boldsymbol{x}', \boldsymbol{x}) u(\boldsymbol{x})$. Because $\bar{u}(b\boldsymbol{x}')$ is an average over $u(\boldsymbol{x})$, we must have $\sum_{\boldsymbol{x}} \chi(\boldsymbol{x}', \boldsymbol{x}) = 1$ for any $\boldsymbol{x}'$. Then, we obtain $\sum_{\boldsymbol{x}', \boldsymbol{x}} \chi(\boldsymbol{x}', \boldsymbol{x}) = \sum_{\boldsymbol{x}'} 1 \equiv N'$ (= the number of different $\boldsymbol{x}'$). However, $\sum_{\boldsymbol{x}', \boldsymbol{x}} \chi(\boldsymbol{x}', \boldsymbol{x}) = \sum_{\boldsymbol{x}} A = NA$. Therefore, $AN = N'$, which means $A = b^{-d}$.

Scaling form of correlation functions

- For correlation function in the long-length scale, we have

$$\begin{aligned} G_\mu(|\boldsymbol{x}' - \boldsymbol{y}'|, \boldsymbol{u}') &= \langle \phi'_\mu(\boldsymbol{x}')\phi'_\mu(\boldsymbol{y}')\rangle_{\mathcal{H}(\phi', \boldsymbol{u}')} \\ &\approx b^{2x_\mu}\langle \phi_\mu(\boldsymbol{x})\phi_\mu(\boldsymbol{y})\rangle_{\mathcal{H}(\phi, \boldsymbol{u})} = b^{2x_\mu} G_\mu(|\boldsymbol{x} - \boldsymbol{y}|, \boldsymbol{u}), \end{aligned}$$

  which means $\quad G_\mu(r, \boldsymbol{u}) \approx \dfrac{1}{b^{2x_\mu}} G_\mu\left(\dfrac{r}{b}, \boldsymbol{u}'\right)$

- We will focus on long-range behaviors, which allows us to start from a Hamiltonian which might be obtained after a RGT with a scaling factor so large that all irrelevant fields already have vanished. Also, we consider the case with only one non-zero relevant field, say t:

$$G_\mu(r, t) \approx \frac{1}{b^{2x_\mu}} G_\mu\left(\frac{r}{b}, b^{y_t} t\right).$$

  By choosing $b = r$, we obtain

$$G_\mu(r, t) \approx \frac{1}{r^{2x_\mu}} \tilde{G}_\mu\left(r^{y_t} t\right) \quad \left(\tilde{G}_\mu(x) \equiv G_\mu(1, x)\right)$$

Critical exponents ν and η

- Let us see what we can deduce from the scaling form

$$G_\mu(r,t) \approx \frac{1}{r^{2x_\mu}} \tilde{G}_\mu \left(t r^{y_t}\right) = \frac{1}{r^{2x_\mu}} \tilde{G}_\mu \left(\left(\frac{r}{t^{-1/y_t}}\right)^{y_t}\right)$$

- First, by comparing it with the defining equation of the correlation length, $G_\mu(r,t) \sim r^{-\omega} e^{-r/\xi(t)}$ we can derive

$$\xi(t) \sim t^{-\frac{1}{y_t}} \quad \Rightarrow \quad \nu = \frac{1}{y_t}.$$

- Second, by taking the limit $t \to 0$,

$$G_\mu(r, t=0) \approx \frac{1}{r^{2x_\mu}} \tilde{G}_\mu(0) \tag{6}$$

which means (because of the definition of η_μ)

$$d - 2 + \eta_\mu = 2x_\mu$$

Order parameters and critical exponent β

- Consider the expectation value of a scaling field ϕ_μ,

$$m_\mu(\boldsymbol{u}) \equiv \langle \phi_\mu(\boldsymbol{x}) \rangle_{\boldsymbol{u}} \approx \langle b^{-x_\mu} \phi'_\mu(\boldsymbol{x}') \rangle_{\boldsymbol{u}'} = b^{-x_\mu} m_\mu(\boldsymbol{u}').$$

It follows that $m_\mu(\mathbf{0}) = 0$ if $x_\mu > 0$, which we assume below.

- Suppose that spontaneous magnetization, though our discussion extends to other quantities, exists (i.e., $\langle \phi_\mu \rangle > 0$) slightly away from the critical point.

$$m_\mu(t) \approx b^{-x_\mu} m_\mu(b^{y_t} t).$$

- By choosing $b = |t/t_0|^{-1/y_t}$, with t_0 being any constant, we obtain

$$m_\mu(t) \propto t^{\frac{x_\mu}{y_t}},$$

Thus, the critical exponent β is related to the scaling dimensions, i.e.,

$$\beta = \frac{x_\mu}{y_t}.$$

Gaussian model and Gaussian fixed point

- Consider the Gaussian model:

$$\mathcal{H}_a(\boldsymbol{\phi}|\rho, t) \equiv \int_a^L d^d\boldsymbol{x}\left(\rho(\nabla\phi_{\boldsymbol{x}})^2 + t\phi_{\boldsymbol{x}}^2 - h\phi_{\boldsymbol{x}}\right)$$
$$= \int_{\pi/L}^{\pi/a} \frac{d^d\boldsymbol{k}}{(2\pi)^d}(\rho k^2 + t)\phi_{\boldsymbol{k}}^2 - h\phi_{\boldsymbol{0}}.$$

(∗ The lower-bound of the integrals symbolically specifies the short-range cutoff.)

- We will apply the RG transformation:

Partial Trace: $\mathcal{H}_a(\phi|\rho, t, h) \to \mathcal{H}_{ba}(\tilde{\phi}|\tilde{\rho}, \tilde{t}, \tilde{h})$
Rescaling: $\mathcal{H}_{ba}(\tilde{\phi}|\tilde{\rho}, \tilde{t}, \tilde{h}) \to \mathcal{H}_a(\phi'|\rho', t', h')$

Partial trace of short-range fluctuation

- (Partial trace) $\mathcal{H}_a(\phi|\rho, t, h) \to \mathcal{H}_{ab}(\tilde{\phi}|\tilde{\rho}, \tilde{t}, \tilde{h})$

Since each wave-number component is independent from the others, the summation over $\phi_{\boldsymbol{k}}$ for $|\boldsymbol{k}| > \pi/2a$ results simply in a multiplicative constant:

$$e^{-\mathcal{H}_{ba}(\tilde{\phi}|\tilde{\rho},\tilde{t},\tilde{h})} \equiv \int d\{\phi_{\boldsymbol{k}}\}_{|\boldsymbol{k}|>\frac{\pi}{ba}} e^{-\int_{\pi/L}^{\pi/a}\frac{d^d\boldsymbol{k}}{(2\pi)^d}\left(\rho k^2+t\right)\phi_{\boldsymbol{k}}^2 + h\phi_{\boldsymbol{0}}}$$
$$\sim e^{-\int_{\pi/L}^{\pi/ba}\frac{d^d\boldsymbol{k}}{(2\pi)^d}\left(\rho k^2+t\right)\phi_{\boldsymbol{k}}^2 + h\phi_{\boldsymbol{0}}},$$

or $$\mathcal{H}_{ba}(\tilde{\phi}|\tilde{\rho}, \tilde{t}) = \int_{\pi/L}^{\pi/ba} \frac{d^d\boldsymbol{k}}{(2\pi)^d}\left(\rho k^2 + t\right)\phi_{\boldsymbol{k}}^2 - h\phi_{\boldsymbol{0}}.$$

In short, the partial trace amounts to

$$\tilde{\phi}_{\boldsymbol{k}} = \phi_{\boldsymbol{k}} \quad \left(\text{for } |\boldsymbol{k}| < \frac{\pi}{ba}\right), \quad (\tilde{\rho}, \tilde{t}, \tilde{h}) = (\rho, t, h).$$

Rescaling

- (Rescaling) $\mathcal{H}_{ba}(\tilde{\phi}|\tilde{\rho},\tilde{t}) \to \mathcal{H}_a(\phi'|\rho',t')$ $\left(\boldsymbol{k}' \equiv b\boldsymbol{k},\ \phi'_{\boldsymbol{k}'} = b^{-\omega}\tilde{\phi}_{\boldsymbol{k}}\right)$

$$\mathcal{H}_a(\phi'|\rho',t',h') = \int_{b\pi/L}^{\pi/a} \frac{d^d\boldsymbol{k}'}{(2\pi)^d} b^{-d}\left(\rho b^{-2}k'^2 + t\right) b^{2\omega}{\phi'_{\boldsymbol{k}'}}^2 - h\phi_{\mathbf{0}}$$

$$= \int_{b\pi/L}^{\pi/a} \frac{d^d\boldsymbol{k}'}{(2\pi)^d} b^{-(d+2)+2\omega}\left(\rho\boldsymbol{k}'^2 + b^2 t\right){\phi'_{\boldsymbol{k}'}}^2 - b^{\omega}h\phi'_{\mathbf{0}}$$

The exponent ω must be $\frac{d+2}{2}$ to make ρ unchanged by RGT. Then,

$$\mathcal{H}_a(\phi'|\rho',t',h') = \int_{b\pi/L}^{\pi/a} \frac{d^d\boldsymbol{k}'}{(2\pi)^d}\left(\rho k'^2 + t'\right){\phi'_{\boldsymbol{k}'}}^2 - h'\phi'_{\mathbf{0}}$$

with $t' \equiv b^2 t$, and $h' \equiv b^{y_h}h$. $(y_h = \omega = (d+2)/2)^*$

$y_t = 2$ and $y_h = \frac{d+2}{2}$ (Gaussian model)

* This means $\phi'_{k'} = b^{-y}\phi_k$. This is consistent with $\phi'_{x'} = b^x\phi_x$, as we see later.

Summary of RGT of Gaussian model

- By RG transformation,

$$\mathcal{H}_a(\phi|\rho,t,h) = \int_{b\pi/L}^{\pi/a} \frac{d^d\boldsymbol{k}}{(2\pi)^d}\left(\rho k^2 + t\right){\phi_{\boldsymbol{k}'}}^2 - h\phi_{\mathbf{0}}$$

is transformed into

$$\mathcal{H}_a(\phi'|\rho',t',h') = \int_{b\pi/L}^{\pi/a} \frac{d^d\boldsymbol{k}'}{(2\pi)^d}\left(\rho' k'^2 + t'\right){\phi'_{\boldsymbol{k}'}}^2 - h'\phi'_{\mathbf{0}}$$

with

$$\boldsymbol{k}' = b\boldsymbol{k},\ \phi'_{\boldsymbol{k}'} = b^{-y_h}\phi_{\boldsymbol{k}},\ \rho' = \rho,\ t' = b^{y_t}t,\ h' = b^{y_h}h \tag{7}$$

with

$$y_t \equiv 2 \quad \text{and} \quad y_h \equiv \frac{d+2}{2}. \tag{8}$$

RGT on $\phi_{\boldsymbol{x}}$

- We saw $x_\mu = d - y_\mu$ in general, its direct derivation in the case of Gaussian model clarifies the meaning of RGT.
- Considering the Fourier components of $\phi'_{\boldsymbol{x}'}$,

$$\phi'(\boldsymbol{x}') = L'^{-d} \sum_{k'}^{\pi/a} e^{i\boldsymbol{k}'\boldsymbol{x}'} \phi'_{\boldsymbol{k}'} = b^d L^{-d} \sum_{\boldsymbol{k}}^{\pi/ab} e^{i\boldsymbol{k}\boldsymbol{x}} b^{-y} \phi_{\boldsymbol{k}}$$
$$= b^{d-y} L^{-d} \sum_{\boldsymbol{k}}^{\pi/ab} e^{i\boldsymbol{k}\boldsymbol{x}} \phi_{\boldsymbol{k}} = b^x [\phi(\boldsymbol{x})]_{k<\frac{\pi}{ab}}$$

- Here, $[\phi(\boldsymbol{x})]_{k<k^*} \equiv L^{-d} \sum_{\boldsymbol{k}}^{k^*} e^{i\boldsymbol{k}\boldsymbol{x}} \phi_{\boldsymbol{k}}$ is something one obtains after filtering out the short wave-length part $(k > k^*)$ from $\phi(\boldsymbol{x})$. Therefore, $\phi(\boldsymbol{x})$ and $[\phi(\boldsymbol{x})]_{k<k^*}$ are identical in long-range behaviors.

ν and η of Gaussian model

- In general,

$$\nu = \frac{1}{y_t}, \quad d - 2 + \eta_\mu = 2x_\mu$$

- For the Gaussian model, we have derived

$$y_t = 2 \quad \text{and} \quad y_h = \frac{d+2}{2} \quad \left(x_h = \frac{d-2}{2} \right)$$

- Therefore, for the Gaussian model

$$\nu = \frac{1}{2} \quad \text{and} \quad \eta = 0.$$

Exercise 7.1: Show that the critical exponent γ_μ that describes the temperature-dependence of the susceptibility,
$\chi_\mu \equiv \partial\langle\phi_\mu(\boldsymbol{x})\rangle/\partial u_\mu \propto t^{-\gamma_\mu}$, is related to the scaling dimensions/eigenvalues as $\gamma_\mu = \dfrac{y_\mu - x_\mu}{y_t} = \dfrac{2y_\mu - d}{y_t}$.

$$\begin{aligned}\chi(u_t) &\sim \sum_{\boldsymbol{y}}\langle\phi_\mu(\boldsymbol{x})\phi_\mu(\boldsymbol{y})\rangle = \sum_{\boldsymbol{y}} b^{-2x_\mu}\langle\phi'_\mu(\boldsymbol{x}/b)\phi'_\mu(\boldsymbol{y}/b)\rangle \\ &= \sum_{\boldsymbol{y}'} b^{d-2x_\mu}\langle\phi'_\mu(\boldsymbol{x}/b)\phi'_\mu(\boldsymbol{y}')\rangle = b^{2y_\mu - d}\chi(u'_t) \sim b^{2y_\mu - d}\chi(b^{y_t}u_t)\end{aligned}$$

By taking $b \sim u^{-\frac{1}{y_t}}$, we obtain $\chi(u) \sim u^{-\frac{2y_\mu - d}{y_t}}$, which means $\gamma_\mu = \dfrac{2y_\mu - d}{y_t}$.

For example, γ_h for the Gaussian model is $\gamma_h = \dfrac{2y_h - d}{y_t} = 1$.

Exercise 7.2: Consider a system for which the susceptibility χ_μ diverges as one approaches the critical point keeping the condition $u_\mu = 0$. Does application of infinitesimal field u_μ qualitatively change the critical properties? Can we say the opposite, i.e., that the field does not essentially change the nature of the transition whenever $\chi_\mu < \infty$?

Exercise 7.3: In the rescaling of the Gaussian model, we fixed y_h so that the ρ would not change. In principle, we should be able to obtain some RGT by fixing other parameters in stead of ρ. What would we have obtained, for example, if we had fixed t rather than ρ?