Lecture 6: Simple Real-Space RG Transformations

Naoki KAWASHIMA

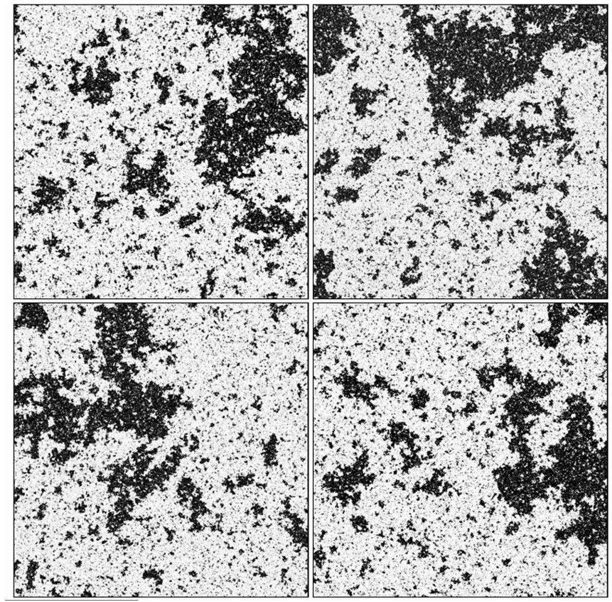
ISSP, U. Tokyo

May 27, 2024

In this lecture, we see

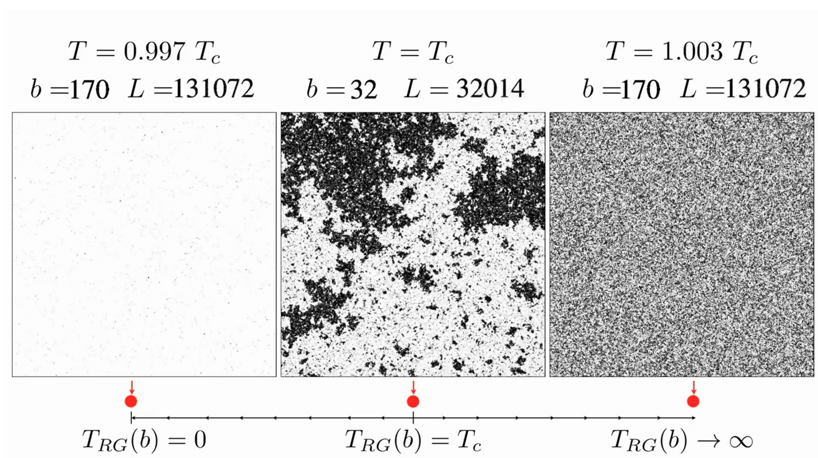
- The fixed point of the RG transformation is important in understanding our world.
- Real-space renormalization group transformation is generally impossible to carry out in dimension higher than 1. Therefore, it requires some approximation.
- A decimation-based RG can be approximately done by a method proposed by Migdal and Kadanoff.
- Generally, the fixed point of RG transformation (RGT) represents the critical point.
- MKRG produces a non-trivial evaluates of critical exponents.
- However, they do not generally agree with the correct values, and it is not obvious how to systematically improve the approximation.

Critical point is scale-invariant



“https://youtu.be/fi-g2ET97W8” by Douglas Ashton

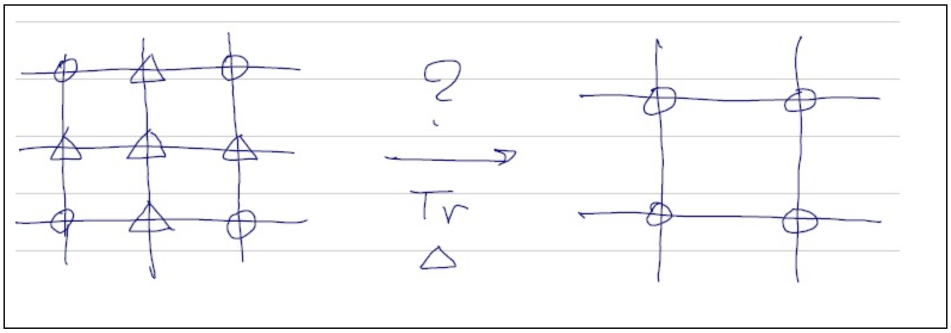
Coarse-graining flow



“https://youtu.be/MxRddFrEnPc” by Douglas Ashton

“Critical point” = “Fixed point of RG transformation”

1D was easy. Can we do the same in 2D case?



RG is more tricky for $d > 1$

- Consider, for example, coarse-graining by decimation.

$$\tilde{S}_{\boldsymbol{x}} \equiv S_{\boldsymbol{x}} \quad \text{for } \boldsymbol{x} \in \Omega' \equiv \{(2ma, 2na) | m, n = 0, 1, 2, \cdots, L/2\}$$

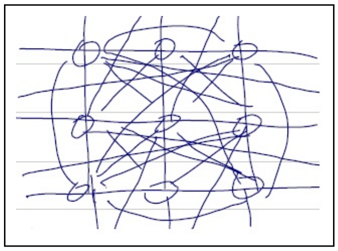
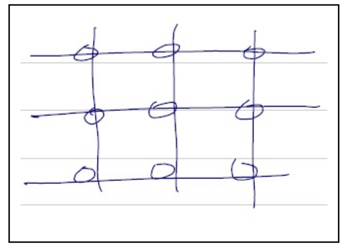
- The partial trace can be taken (at least formally) as

$$e^{-\tilde{\mathcal{H}}_{2a}(\tilde{\boldsymbol{S}}, \tilde{K})} \equiv \operatorname*{Tr}_{\{S_{\boldsymbol{x}}\}_{\boldsymbol{x} \in \Omega \setminus \Omega'}} e^{-\mathcal{H}_a(\boldsymbol{S}, K)}$$

- There are paths that connect two remaining spins, say $S_{\boldsymbol{r}}$ and $S_{\boldsymbol{r}'}$ $(\boldsymbol{r}, \boldsymbol{r}' \in \Omega')$, through $\Omega \setminus \Omega'$. Tracing out the spins along the path give rise to the long-range interaction between $S_{\boldsymbol{r}}$ and $S_{\boldsymbol{r}'}$.
- There are n-body $(n > 2)$ interactions among the remaining spins, because, for example, $\sum_{S_0} (1 + tS_0 S_1)(1 + tS_0 S_2)(1 + tS_0 S_3)$ $(1 + tS_0 S_4)$ contains the term like $t^4 S_1 S_2 S_3 S_4$.
- As a result, unlike the 1D case, the renormalized Hamiltonian is too complicated to deal with.

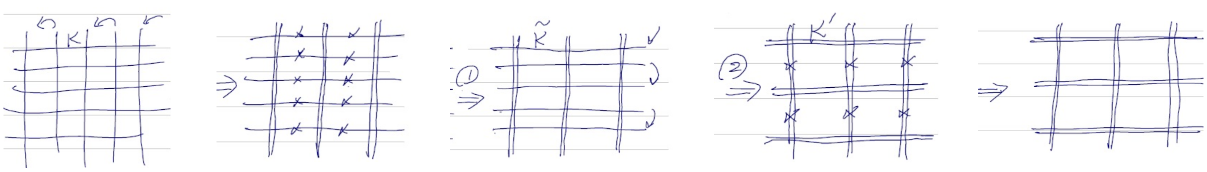
$d > 1$ needs approximation

The renormalized Hamiltonian

is more like  than  .

We can carry out the RG mapping only approximately.

Migdal-Kadanoff approximation for 2D Ising model



- Bunch up two vertical lines.
- Partial trace of spins ($\times$) on horizontal bonds. (①: $\mathrm{th}\,\tilde{K} = \mathrm{th}^2\,K$)
- Bunch up two horizontal lines. (②: $\acute{K} = 2\tilde{K}$)
- Partial trace of intermediate spins ($\times$) on vertical bonds.
- Trivial re-scaling (do nothing on $\acute{K}$).

simple Migdal-Kadanoff

$$t' = \mathrm{th}(2\,\mathrm{ath}(t^2)) \left(= \frac{2t^2}{1+t^4} \right) \quad (t \equiv \mathrm{th}\,K,\ t' \equiv \mathrm{th}\,\acute{K})$$

Effect of RGT on correlation function

- Consider a decimation-based RG transformation with scaling factor b

  $$t' = R_b(t),$$

  e.g., $R_2(t) = \mathrm{th}(2\,\mathrm{ath}(t^2))$ for MKRG.

- For the decimation-based RG $^{(*)}$, the renormalized spin is simply the original spin at the corresponding position, i.e., $S'(r') = S(r)$ $(r' \equiv r/b)$. Therefore, the correlation function must satisfy

  $$C(b^{-1}r|t') \equiv \langle S'(b^{-1}r)S'(0)\rangle_{t'} = \langle S(r)S(0)\rangle_t \equiv C(r|t). \qquad (1)$$

- Assuming that $C(r|t)$ decays asymptotically like $e^{-r/\xi(t)}$, (1) yields $b^{-1}r/\xi(t') = r/\xi(t)$. Thus, we obtain

  $$\xi(t') = b^{-1}\xi(t). \qquad (2)$$

$^{(*)}$ Essentially the same argument can be made for more general RG, but there is no need for additional argument about the nature of the renormalized correlation function/length for the decimation-based RG as can be seen below.

RG fixed point and $y_t = 1/\nu$

- The RG fixed-point t_c is defined by $t_c = R_b(t_c)$.
- The RGT amplifies the 'deviation' from the fixed-point as

  $$\delta t \to \delta t' = t' - t_c = R_b(t_c + \delta t) - t_c \approx R_b'(t_c)\delta t$$

- Therefore, (2) means $\xi(t_c + R_b'\delta t) \approx b^{-1}\xi(t_c + \delta t)$.
- Since the exponent ν is defined by $\xi \propto (\delta t)^{-\nu}$,

  $$(R_b'\delta t)^{-\nu} = b^{-1}(\delta t)^{-\nu} \quad \to \quad R_b'^{-\nu} = b^{-1}$$

  $$\to \quad y_t \equiv \frac{1}{\nu} = \frac{\log R_b'(t_c)}{\log b} \qquad (3)$$

Derivatives of RG transformation are critical exponents.

RG fixed point and $y_t = 1/\nu$ (numerical estimates)

- For the Migdal-Kadanoff RGT for 2D Ising model, we have

$$R_2(t_c) = \frac{2t_c^2}{1+t_c^4} = t_c \to \ t_c = 0.54368\cdots$$

(cf: $t_c^{\mathrm{exact}} = \sqrt{2} - 1 = 0.4142\cdots$)

- With some arithmetic, we can get

$$R_2'(t_c) = \frac{2(1-t_c)}{t_c} \approx 1.676$$

$$\to y_t \equiv 1/\nu \approx \log 1.676 / \log 2 \approx 0.747$$

(cf: $y_t^{\mathrm{exact}} = 1$, $y_t^{\mathrm{mean\ field}} = 2$)

Not bad, but ad-hoc (not obvious how to improve).

Infinitesimal MKRG

- The MKRG mapping with the scaling factor b can be summarized as

$$t' = R_2(t) \equiv \mathrm{th}(2\,\mathrm{ath}(t^2)).$$

- This can be generalized formally to general integer $b > 1$ as

$$t' = R_b(t) \equiv \mathrm{th}(b\,\mathrm{ath}(t^b)).$$

  Although the corresponding operation cannot be defined for non-integer b, let us assume that it is still meaningful.

- After all, "bunching-up" two lines to one by one step might be too crude. It may become less harmful if we bunch-up as small number of lines as possible, i.e., taking $b = 1 + \lambda$ where $0 < \lambda \ll 1$

Does this infinitesimal RG still yield sensible results?

Infinitesimal RG (general argument)

- In general, suppose some RG transformation $t' = R_b(t)$ with continuous scaling factor $b = 1 + \lambda$.
- Using the notation $\dot{f} \equiv \partial f / \partial b$, the critical point is determined by

$$t_c = R_{1+\lambda}(t_c) = R_1(t_c) + \lambda \dot{R}_1(t_c) \Rightarrow \dot{R}_1(t_c) = 0.$$

($\dot{R}_1(t)$ is called "beta function" and the symbol $\beta(t)$ is often used.)

- Using $R_1(t) = t$ (therefore $R_1'(t) = 1$, and $R_1''(t) = 0$), in the lowest order in λ, the scaling dimension y_t can be obtained by

$$y_t(1+\lambda) = \frac{\log\left(R'_{1+\lambda}(t_c(1+\lambda))\right)}{\log(1+\lambda)} \quad \text{(See Eq.(3))}$$

$$\approx \frac{1}{\lambda} \log\Big(R_1'(t_c(1)) + R_1''(t_c(1))\Delta t_c + \dot{R}_1'(t_c(1))\lambda\Big) \approx \dot{R}_1'(t_c(1)).$$

$\dot{R}_1(t_c) = 0$ and $y_t = \dot{R}_1'(t_c)$

Infinitesimal MKRG (numerical estimates)

- For $b = 1 + \lambda$ ($\lambda \ll 1$), defining $t \equiv \operatorname{th} K$, we obtain

$$t' = R_b(t) = \operatorname{th}(b \operatorname{ath} t^b) \approx t + \lambda \rho(t)$$

$$\rho(t) \equiv \left.\frac{\partial R_b(t)}{\partial b}\right|_{b\to 1} = (1-t^2) \operatorname{ath} t + t \log t$$

- The critical point $t = t_c$ is determined by $\rho(t_c) = 0$, which has the solution

$$t_c = \sqrt{2} - 1 \quad \text{(Exactly agrees with the correct value!)}$$

- As for y_t, we have

$$y_t = \rho'(t_c) = 2 + \sqrt{2}\log(\sqrt{2} - 1) = 0.753549\cdots,$$

slightly closer to $y_t^{\text{exact}} = 1$ than the simple MKRG with $b = 2$.

Better, but still ad-hoc (not obvious how to further improve).

Exercise 6.1: Try the idea of MKRG (i.e., bunching up and trace over intermediate spins) on the Ising model in higher dimensions ($d > 2$).