Lecture 5: Introduction to Renormalization Group

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In this lecture we see ...

- There are cases where we can rely on the mean-field theory even for the critical behavior. (Ginzburg criterion)
- However, in low dimensions including $d = 3$, the mean-field theory is not self-consistent concerning the critical phenomena.
- We can define the renormalization group (RG) transformation, and if we can calculate its result, we would be able to discuss the critical properties of the system.
- For 1D Ising model, we can carry out the RG transformation, which yields correct critical behavior.

When can MF be valid? — Ginzburg criterion

- First, we will elucidate the meaning of the asymptotic validity and draw a general criterion.
- Then, we will check whether the mean-field theory satisfies the criterion in a self-consistent way.
- We will find that it is indeed self-consistent in some cases, but not in general. (Ginzburg criterion)

Asymptotic validity of MF approximation

- Consider a system just below the critical temperature, where there is a finite but small spontaneous magnetization.
- The mean-field (MF) description should be valid when the relative fluctuation is negligible, i.e., $\delta\phi_{\boldsymbol{r}} \ll \langle\phi_{\boldsymbol{r}}\rangle$
- Typically, this condition is **not** satisfied at the scale of lattice constant, e.g., for the Ising model, $\langle\phi_{\boldsymbol{r}}\rangle \approx 0$ and $\delta\phi_{\boldsymbol{r}} = \sqrt{\langle\delta\phi_{\boldsymbol{r}}^2\rangle} \approx 1$.
- However, the MF description may still be qualitatively correct at larger length-scales b, greater than the lattice constant, a, and smaller than the correlation length, ξ, relevant to the critical behavior.
- So, we consider the local average of ϕ, i.e., $\bar{\phi}_{\boldsymbol{R}} \equiv \frac{1}{b^d}\sum_{\boldsymbol{r}\in\Omega_b(\boldsymbol{R})}\phi_{\boldsymbol{r}}$ over the cluster of size b.
- The condition for asymptotic validity of MF is that, for any λ $(0 < \lambda < 1)$, for clusters of size $b \equiv \lambda\xi$, the ratio $\delta\bar{\phi}_{\boldsymbol{R}}/\langle\bar{\phi}_{\boldsymbol{R}}\rangle$ converges to zero as we approach T_c from below ($\xi \to \infty$).

Self-consistency of mean-field approximation

- For $\langle\bar{\phi}\rangle$, below T_c, we have $\langle\bar{\phi}\rangle^2_{\mathrm{MF}} \sim m^2 \sim \frac{|\Delta t|}{u} \sim \frac{\rho}{u\xi^2}$
- For the amplitude of the fluctuation, we have

$$\langle(\delta\bar{\phi})^2\rangle_{\mathrm{MF}} = \left(\frac{a}{b}\right)^{2d} \sum_{\boldsymbol{r},\boldsymbol{r}'\in\Omega_b(\boldsymbol{R})} \langle\delta\phi_{\boldsymbol{r}'}\delta\phi_{\boldsymbol{r}}\rangle \lesssim \frac{A(\lambda)}{\rho\xi^{d-2}} \quad (* \text{ see supplement})$$

where A depends on ξ only through $\lambda \equiv b/\xi$.

- It follows that, for any λ, the ratio

$$\langle\delta\bar{\phi}^2\rangle_{\mathsf{MF}}/\langle\bar{\phi}\rangle^2_{\mathsf{MF}} \sim A(\lambda)u\rho^{-2}\xi^{4-d}$$

always converges to 0 if $d > 4$, and diverges if $d < 4$ as $\xi \to \infty$.

Ginzburg criterion (Upper critical dimension)

The MF approximation to ϕ^4 model is asymptotically correct if $d > 4$, and invalid if $d < 4$.

Supplement: MF estimate of fluctuation

In Lecture 3, we saw

$$\langle\delta\phi_{\boldsymbol{r}'}\delta\phi_{\boldsymbol{r}}\rangle \sim \frac{1}{\rho}\frac{\kappa'^{d-2}}{(\kappa' r)^{\frac{d-1}{2}}} e^{-\kappa'|\boldsymbol{r}'-\boldsymbol{r}|}, \qquad \left(\kappa' = \frac{1}{\xi} \approx \sqrt{-\Delta t}\right)$$

from which we obtain

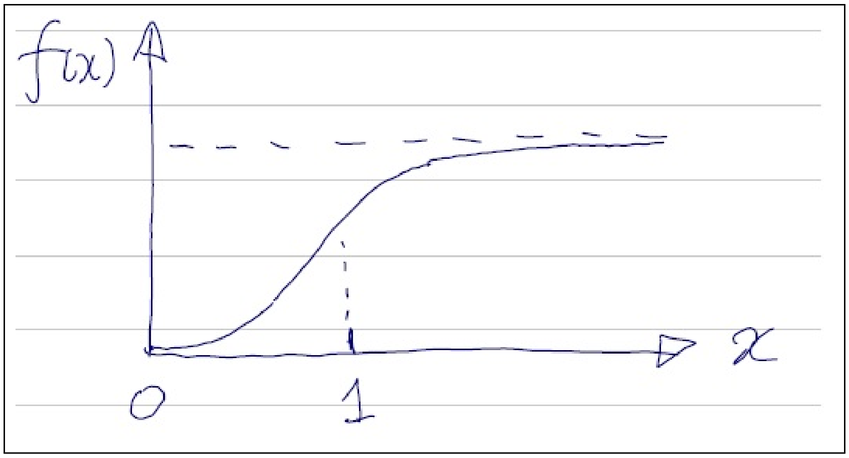
$$\langle(\delta\bar{\phi})^2\rangle = \left(\frac{a}{b}\right)^{2d} \sum_{\boldsymbol{r},\boldsymbol{r}'\in\Omega_b(\boldsymbol{R})} \langle\delta\phi_{\boldsymbol{r}'}\delta\phi_{\boldsymbol{r}}\rangle \sim \left(\frac{a}{b}\right)^{d} \sum_{\Delta\boldsymbol{r}} \frac{\rho^{-1}\kappa'^{d-2}}{(\kappa'|\Delta\boldsymbol{r}|)^{\frac{d-1}{2}}} e^{-\kappa'|\Delta\boldsymbol{r}|}$$

$$\sim \frac{1}{b^d}\int_0^b dr\, r^{d-1} \frac{\rho^{-1}\kappa'^{d-2}}{(\kappa' r)^{\frac{d-1}{2}}} e^{-\kappa' r} \sim \frac{1}{b^d}\frac{1}{\rho\kappa'^2}\int_0^{\kappa' b} dx\, x^{\frac{d-1}{2}} e^{-x}$$

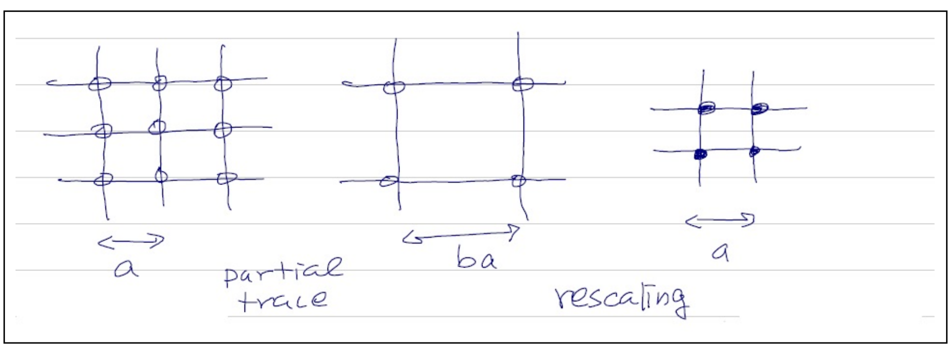
$$\sim \frac{f(\kappa' b)}{\rho\kappa'^2 b^d} \qquad \left(f(x) \sim \begin{cases} x^{\frac{d+1}{2}} & (x \ll 1) \\ f_\infty \ \text{(a dimension-less constant)} & (x \gg 1) \end{cases} \right)$$

$$\sim \frac{\kappa'^{2-d}}{\rho} \times \frac{f(\kappa' b)}{(\kappa' b)^d} = \frac{A(\lambda)}{\rho\xi^{d-2}}. \qquad \left(A(\lambda) \equiv \frac{f(\lambda)}{\lambda^d},\ \lambda \equiv \frac{b}{\xi}\right)$$

Supplement: MF estimate of fluctuation



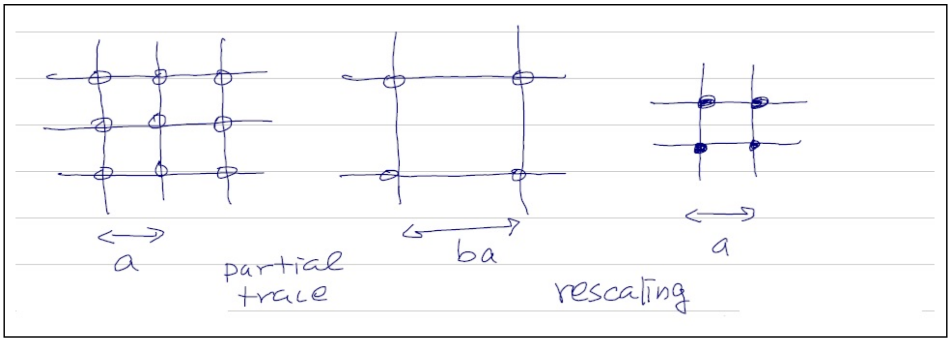
General renormalization group (RG) transformation



- In the derivation of the Ginzburg criterion, we introduced the coarse-graining transformation as a Gedankenexperiment.
- The RG transformation consists of two steps: (i) coarse-graining and (ii) rescaling. Schematically,

$$\mathcal{H}_a(S \,|\, \boldsymbol{K}, L) \xrightarrow{\text{(i)}} \mathcal{H}_{ab}(\tilde{S} \,|\, \tilde{\boldsymbol{K}}, L) \xrightarrow{\text{(ii)}} \mathcal{H}_a(\acute{S} \,|\, \acute{\boldsymbol{K}}, b^{-1}L)$$

General Renormalization Group Transformation



- In the coarse-graining step, we define coarse-grained field and carry out the configuration-space summation of the partition function, with the constraint imposed by the coarse-grained fields.
- In the rescaling step, we redefine the length-scale and the field variables by multiplying them with scaling factors so that the effective Hamiltonian may be the same form as the original one.

Coarse-graining

In the coarse-graining step of the RG procedure, we first define "coarse-grained field", $\tilde{S}_{\boldsymbol{R}}$, which is defined in terms of $S_{\boldsymbol{r}}$ in the neighborhood of $\boldsymbol{R}$, i.e., $\tilde{S}_{\boldsymbol{R}} = \Sigma(\{S_{\boldsymbol{r}}\}_{\boldsymbol{r}\in\Omega_b(\boldsymbol{R})})$, with some function $\Sigma(\cdots)$. More formally,

$$e^{-\mathcal{H}_a(S|\boldsymbol{K},L)} \to e^{-\mathcal{H}_{ab}(\tilde{S}|\tilde{\boldsymbol{K}},L)} \equiv \sum_S \Delta(\tilde{S}\,|\Sigma(S))e^{-\mathcal{H}_a(S|\boldsymbol{K},L)},$$

where $\boldsymbol{K}$ is a set of parameters such as $\boldsymbol{K} \equiv (\beta, H)$.

Example 1 (Ising chain with $b = 3$)

$$\Sigma(S_1, S_2, S_3) = S_2 \quad \text{(Simple decimation)}$$
$$\Sigma(S_1, S_2, S_3) = (S_1 + S_2 + S_3)/3 \quad \text{(Local Average)}$$
$$\Sigma(S_1, S_2, S_3) = \mathrm{sign}(S_1 + S_2 + S_3) \quad \text{(Majority rule)}$$

Example: Coarse-graining of Ising chain ($b = 2$)

- Consider the Ising model of size $L \equiv 2^g$ in one dimension.

$$\mathcal{H}_a(S|\boldsymbol{K}, L) = -K \sum_{i=0}^{L-1} S_i S_{i+1} - h \sum_{i=0}^{L-1} S_i \qquad (\boldsymbol{K} \equiv (K, h))$$

- For even L, let us adopt the decimation for the coarse-graining:

$$\tilde{S}_i = S_i \quad (\text{for } i = 0, 2, 4, \cdots, L-2)$$

- Then, $e^{-\mathcal{H}_{2a}(\tilde{S}|\tilde{K},L)} = \sum_{S_1,S_3,\cdots,S_{L-1}} e^{-\mathcal{H}_a(S|K,L)}$. For $h = 0$ we have

$$e^{-\mathcal{H}_{2a}(\tilde{S}|\tilde{K},L)} = \sum_{S_1} e^{K(S_0+S_2)S_1} \sum_{S_3} e^{K(S_2+S_4)S_3} \cdots \sum_{S_{L-1}} e^{K(S_{L-2}+S_0)S_{L-1}}$$

$$\sim e^{\tilde{K}S_0S_2} e^{\tilde{K}S_2S_4} \cdots e^{\tilde{K}S_{L-2}S_0} \sim e^{-\mathcal{H}_{2a}(\tilde{S}|\tilde{K},L)} \quad (\mathrm{th}\, \tilde{K} \equiv (\mathrm{th}\, K)^2)$$

Example: Rescaling of Ising chain ($b = 2$)

- Let us use $t \equiv \mathrm{th}\, K$ instead of K for the parameter. Then, the effect of the coarse-graining on t is

$$\tilde{t} = t^2.$$

- The rescaling in the present case is simply

$$\acute{\boldsymbol{r}} \equiv \boldsymbol{r}/2, \quad \acute{S}_{\acute{\boldsymbol{r}}} \equiv \tilde{S}_{\boldsymbol{r}}, \quad \text{and} \quad \acute{t} \equiv \tilde{t}.$$

- Together with the coarse-graining, we obtain the whole RG transformation,

$$\mathcal{H}_a(S|t, L) \xrightarrow[b=2]{RG} \mathcal{H}_a(\acute{S}|\acute{t}, L/2), \quad \text{with} \quad \acute{t} = t^2.$$

Example: Critical exponent ν

- From the whole RG procedure, we can deduce

$$e^{-r/\xi(t)} \sim \langle S_{\boldsymbol{r}} S_{\boldsymbol{0}} \rangle_t = \langle S_{\acute{\boldsymbol{r}}} S_0 \rangle_{\acute{t}} \sim e^{-\acute{r}/\xi(\acute{t})}$$

- Because $\acute{r} = r/2$,

$$\xi(t) = 2\xi(\acute{t}) \quad \left(\acute{t} = t^2\right).$$

- Since $\acute{t} = t^2$, if we define $g \equiv -\log t$, the correlation length as a function of g would satisfy

$$\xi(g) = 2\xi(2g).$$

- From this, we can obtain $\xi(g)$ upto a constant factor,

$$\xi(g) \sim \frac{1}{g} \qquad \Rightarrow \nu = 1 \quad \text{(Exact!)}$$

Exercise 5.1: By solving the 1D Ising model, compute the correlation function $G(r) \equiv \langle S_r S_0 \rangle$ and the correlation length ξ. Verify $\xi \propto g^{-1}$ where $g \equiv -\log \operatorname{th} K$. (Hint: The correlation function can be expressed as

$$\langle S_r S_0 \rangle = \operatorname{Tr}\left(T^{L-r} \sigma T^r \sigma\right) / \operatorname{Tr}\left(T^L\right)$$

where T is a 2×2 matrix defined as $T_{S',S} \equiv e^{KS'S}$ and σ is another 2×2 matrix defined as $\sigma \equiv \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right)$.)

The matrices T and σ can be diagonalized as

$$T = \begin{pmatrix} e^K & e^{-K} \\ e^{-K} & e^K \end{pmatrix} = U \begin{pmatrix} 2\operatorname{ch} K & 0 \\ 0 & 2\operatorname{sh} K \end{pmatrix} U, \quad \sigma = U \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U,$$

where $U \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Therefore, the correlation function, $C(r) \equiv \langle S_r S_0 \rangle$, of a periodic system of length L can be computed as

$$C(r) = \frac{(2\,\mathrm{ch}\,K)^{L-r}(2\,\mathrm{sh}\,K)^r + (2\,\mathrm{sh}\,K)^{L-r}(2\,\mathrm{ch}\,K)^r}{(2\,\mathrm{ch}\,K)^L + (2\,\mathrm{sh}\,K)^L} = \frac{t^r + t^{L-r}}{1 + t^L}$$

with $t \equiv \mathrm{th}\,K$. Therefore, in the limit $r \ll L$, the correlation function behaves like $C(r) = t^r$. From this, we obtain $e^{-1/\xi} = t$, or $\xi = 1/\log(1/t) = 1/g$.

This is identical to what we obtained from the coarse-graining of the 1D Ising chain.