Lecture 3: ϕ^4 Theory

Naoki KAWASHIMA

ISSP, U. Tokyo

April 22, 2024

In this lecture we see ...

- The mean-field theory discussed in the previous section does not tell us about the spatial correlation because the variational Hamiltonian has trivial spatial structure, i.e., zero correlation among fluctuations in spins at different locations.
- In this lecture, starting from the Ising model, we derive ϕ^4 model. While it inherits the same essential properties from the Ising model, it is defined with continuous degrees of freedom in contrast to the Ising spins.
- The advantage of the continuous degrees of freedom is that they allow as to define a variational Hamiltonian that has non-trivial spatial structure, which will be exploited in the next lecture.

ϕ^4 field theory

- We first see a very "hand-waving" derivation of ϕ^4 field theory starting from the Ising model and using the coarse-graining.
- We next see an alternative derivation which looks less hand-waving, based on the Hubbard-Stratonovich transformation.
- Since the ϕ^4 theory is obtained by the coarse-graining of the Ising model, they are supposed to share the same long-range behavior, while they may differ quantitatively for short-range physics.
- In particular, we expect, ϕ^4 model belongs to the same universality class as the Ising model, as has been verified by a number of arguments and numerical calculations.

Coarse-graining

- Let us consider the Ising model on d-dimensional hyper-cubic lattice. (Hereafter, we use symbols like $\boldsymbol{x}$ and $\boldsymbol{y}$, instead of i and j, to specify lattice points.)
- Divide the whole lattice into cells of size ab, where a is the lattice constant and $b \gg 1$, and denote the one located at $\boldsymbol{X}$ as $\Omega_b(\boldsymbol{X})$.
- Consider the "cell average" of spins $\phi_{\boldsymbol{X}} = \left(\frac{1}{b}\right)^d \sum_{\boldsymbol{x} \in \Omega_b(\boldsymbol{X})} S_{\boldsymbol{x}}$
- Consider the coarse-grained Hamiltonian $\tilde{\mathcal{H}}$ defined as

$$e^{-\tilde{\mathcal{H}}(\boldsymbol{\phi})} \equiv \sum_{\boldsymbol{S}} \Delta(\boldsymbol{S}|\boldsymbol{\phi}) e^{-\mathcal{H}(\boldsymbol{S})}$$

  where $\boldsymbol{\phi} \equiv \{\phi_{\boldsymbol{X}}\}$, $\boldsymbol{S} \equiv \{S_{\boldsymbol{x}}\}$, and $\Delta(\boldsymbol{S}|\boldsymbol{\phi})\,(=0,1)$ takes 1 if and only if $\phi_{\boldsymbol{X}}$ equals the cell average of original spins in the cell at every $\boldsymbol{X}$.
- The function $\tilde{\mathcal{H}}$ is a very complicated one. So, we try to construct a simple approximation by intuition.

An intuitive approximation of $\tilde{\mathcal{H}}$

- $\tilde{\mathcal{H}}$ must have two parts: a single-cell part reflecting the physics inside each cell and a multiple-cell part for inter-cell interactions.
- The single-cell part itself consists of two parts: the energy and the entropy. The internal energy tends to align spins parallel to each other, giving rise to $-\phi^2$ term, while the internal entropy favors $\phi \sim 0$ state, producing the terms like $+\phi^2$ and $+\phi^4$.
- For the multiple-cell part, since the total energy should be larger for large spatial inhomogeneity. It would be represented by terms like $(\nabla\phi)^2$, etc, while the odd order terms like $\nabla\phi$ should not appear because of the symmetry of the system.
- Putting these together and including the Zeeman term,

$$\tilde{\mathcal{H}}(\boldsymbol{\phi}) = \sum_{\boldsymbol{X}} \left(\rho|\nabla\phi|^2 + t\phi^2 + u\phi^4 - h\phi\right)$$

($\rho, u > 0$. The sign of t depends on the temperature.)

Derivation by the Hubbard-Stratonovich transformation

$$Z_{\text{Ising}} = \sum_{\boldsymbol{S}} e^{K\sum_{(\boldsymbol{x},\boldsymbol{x}')} S_{\boldsymbol{x}} S_{\boldsymbol{x}'}} \propto \sum_{\boldsymbol{S}} e^{\frac{K}{2}\boldsymbol{S}^{\mathsf{T}} C \boldsymbol{S}} \quad \left(C_{\boldsymbol{x},\boldsymbol{x}'} = \begin{cases} c & (|\boldsymbol{x}-\boldsymbol{x}'|=0) \\ 1 & (|\boldsymbol{x}-\boldsymbol{x}'|=a) \\ 0 & (\text{otherwise}) \end{cases} \right)$$

$$\propto \sum_{\boldsymbol{S}} \int D\boldsymbol{\psi}\, e^{-\frac{1}{2K}\boldsymbol{\psi}^{\mathsf{T}} C^{-1}\boldsymbol{\psi} + \boldsymbol{\psi}^{\mathsf{T}}\boldsymbol{S}} \quad \cdots \text{ HS transformation}$$

$$= \int D\boldsymbol{\psi}\, e^{-\frac{1}{2K}\boldsymbol{\psi}^{\mathsf{T}} C^{-1}\boldsymbol{\psi} + \sum_{\boldsymbol{x}} \log\cosh\psi_{\boldsymbol{x}}} \quad \cdots \text{ trace over } \boldsymbol{S}$$

$$\propto \int D\boldsymbol{\phi}\, e^{-\left(\frac{K}{2}\boldsymbol{\phi}^{\mathsf{T}} C^{-1}\boldsymbol{\phi} - \sum_{\boldsymbol{x}} \log\cosh(K\phi_{\boldsymbol{x}})\right)} \quad \cdots \; \boldsymbol{\phi} \equiv K^{-1}\boldsymbol{\psi}$$

$$= \int D\boldsymbol{\phi}\, e^{-\mathcal{H}_{\text{HS}}(\boldsymbol{\phi})}$$

$$\Rightarrow \quad \mathcal{H}_{\text{HS}}(\boldsymbol{\phi}) = \frac{K}{2}\boldsymbol{\phi}^{\mathsf{T}} C^{-1}\boldsymbol{\phi} - \sum_{\boldsymbol{x}} \log\cosh(K\phi_{\boldsymbol{x}})$$

Relevant part of $\mathcal{H}_{\text{HS}}$

$$\mathcal{H}_{\text{HS}}(\boldsymbol{\phi}) = \frac{K}{2}\boldsymbol{\phi}^{\mathsf{T}} C^{-1}\boldsymbol{\phi} - \sum_{\boldsymbol{x}} \log\cosh(K\phi_{\boldsymbol{x}})$$

For the first term, we can express C^{-1} as $C^{-1} = (cI + \Delta)^{-1} = \frac{1}{c}\left(I - \frac{\Delta}{c} + \cdots\right)$ where Δ is the lattice Laplacian. For $c > z$, the series converges exponentially. So, we take up to the first order to obtain

$$\boldsymbol{\phi}^{\mathsf{T}} C^{-1}\boldsymbol{\phi} \approx \sum_{\boldsymbol{x}} \left(\frac{c-z}{c^2}\phi_{\boldsymbol{x}}^2 + \frac{1}{c^2}(\nabla\phi_{\boldsymbol{x}})^2\right) \qquad \text{(see supplement)}$$

For the second term, by expending it w.r.t. K $\left(\log\cosh(x) \approx x^2/2 - x^4/12\right)$, we obtain $\mathcal{H}_{\phi^4}$ as the relevant part of $\mathcal{H}_{\text{HS}}$ $\left(\text{with } \rho \equiv \frac{K}{2}\frac{c-z}{c^2},\ t \equiv \frac{1}{2}(\frac{K}{c^2} - K^2),\ u \equiv \frac{K^4}{12}\right)$:

$$\mathcal{H}_{\text{HS}} \approx \mathcal{H}_{\phi^4} \equiv \sum_{\boldsymbol{x}} \left(\rho(\nabla\phi)^2 + t\phi^2 + u\phi^4\right) \qquad (1)$$

Though the expansions used here may not be justified by the smallness of variables, they are justified by the renormalization group arguments.

Supplement: Hubbard-Stratonovich transformation

For an arbitrary positive definite symmetric matrix A and a vector $\boldsymbol{B}$, we can show the following,

$$\begin{aligned}
&\int D\boldsymbol{\psi}\, e^{-\frac{1}{2}\sum_{\boldsymbol{x},\boldsymbol{x}'} A_{\boldsymbol{x},\boldsymbol{x}'}\psi_{\boldsymbol{x}}\psi_{\boldsymbol{x}'} + \sum_{\boldsymbol{r}} B_{\boldsymbol{r}}\psi_{\boldsymbol{x}}} \\
&\quad = \int D\boldsymbol{\psi}\, e^{-\frac{1}{2}\boldsymbol{\psi}^{\mathsf{T}} A\boldsymbol{\psi} + \boldsymbol{B}^{\mathsf{T}}\boldsymbol{\psi}} \\
&\quad = \int D\boldsymbol{\xi} |A|^{-1/2} e^{-\frac{1}{2}\boldsymbol{\xi}^{\mathsf{T}}\boldsymbol{\xi} + \boldsymbol{\eta}^{\mathsf{T}}\boldsymbol{\xi}} \quad (\boldsymbol{\xi} \equiv A^{1/2}\boldsymbol{\psi},\ \boldsymbol{\eta} \equiv A^{-1/2}\boldsymbol{B}) \\
&\quad = \int D\boldsymbol{\xi} |A|^{-1/2} e^{-\frac{1}{2}(\boldsymbol{\xi}-\boldsymbol{\eta})^2 + \frac{1}{2}(\boldsymbol{\eta})^2} \\
&\quad = (2\pi)^{\frac{N}{2}} |A|^{-1/2} e^{\frac{1}{2}(\boldsymbol{\eta})^2} = (2\pi)^{\frac{N}{2}} |A|^{-1/2} e^{\frac{1}{2}\boldsymbol{B}^{\mathsf{T}} A^{-1}\boldsymbol{B}}
\end{aligned}$$

By taking KC for A^{-1} and $\boldsymbol{S}$ for $\boldsymbol{B}$,

$$e^{\frac{K}{2}\boldsymbol{S}^{\mathsf{T}} C\boldsymbol{S}} \sim \int D\boldsymbol{\psi}\, e^{-\frac{1}{2K}\boldsymbol{\psi}^{\mathsf{T}} C^{-1}\boldsymbol{\psi} + \boldsymbol{\psi}^{\mathsf{T}}\boldsymbol{S}}$$

Supplement: The matrix Δ and derivatives

In the derivation of the ϕ^4 action, we considered the inverse of $C \equiv cI + \Delta$, i.e., $C^{-1} = \frac{1}{c}\left(I - \frac{\Delta}{c} + \cdots\right)$ where Δ is the lattice connectivity matrix

$$\Delta_{\boldsymbol{x}'\boldsymbol{x}} \equiv \begin{cases} 1 & \text{(if } \boldsymbol{x}' \text{ and } \boldsymbol{x} \text{ are nearest neighbors)} \\ 0 & \text{(othewise)} \end{cases} .$$

We used the follwoing formula:

$$\begin{aligned} \boldsymbol{\phi}^{\mathsf{T}}\Delta\boldsymbol{\phi} &= 2\sum_{(\boldsymbol{x}',\boldsymbol{x})} \phi_{\boldsymbol{x}'}\phi_{\boldsymbol{x}} = \sum_{(\boldsymbol{x}',\boldsymbol{x})} \left(\phi_{\boldsymbol{x}'}^2 + \phi_{\boldsymbol{x}}^2 - (\phi_{\boldsymbol{x}'} - \phi_{\boldsymbol{x}})^2\right) \\ &= \frac{1}{2}\sum_{\boldsymbol{x},\boldsymbol{\delta}} \left(\phi_{\boldsymbol{x}+\boldsymbol{\delta}}^2 + \phi_{\boldsymbol{x}}^2 - (\phi_{\boldsymbol{x}+\boldsymbol{\delta}} - \phi_{\boldsymbol{x}})^2\right) = \sum_{\boldsymbol{x}} \left(z\phi_{\boldsymbol{x}}^2 - (\nabla\phi_{\boldsymbol{x}})^2\right) \end{aligned}$$

where $\boldsymbol{\delta}$ is a vector pointing to nearest neighbors, and $\nabla\phi_{\boldsymbol{x}}$ is the lattice gradient vector. (It would correspond to the regular nabla operator in the limit of fine space discretization.)

The meaning of ϕ in the HS derivation

Remembering the HS transformation, for any real symmetric matrix A,

$$\begin{aligned} \langle \boldsymbol{S}^{\mathsf{T}} A \boldsymbol{S}\rangle_{\text{Ising}} &= Z_{\text{Ising}}^{-1} \frac{\partial}{\partial\eta} \sum_{\boldsymbol{S}} e^{\frac{K}{2}\boldsymbol{S}^{\mathsf{T}}(C + \frac{2\eta}{K}A)\boldsymbol{S}} \Big|_{\eta\to 0} \\ &= Z_{\text{HS}}^{-1} \int D\boldsymbol{\phi}\, \frac{\partial}{\partial\eta} e^{-\left(\frac{K}{2}\boldsymbol{\phi}^{\mathsf{T}}(C+\frac{2\eta}{K}A)^{-1}\boldsymbol{\phi} - \sum_{\boldsymbol{x}}\log\cosh(K\phi_{\boldsymbol{x}})\right)} \Big|_{\eta\to 0} \\ &= Z_{\text{HS}}^{-1} \int D\boldsymbol{\phi}\, \frac{\partial}{\partial\eta} e^{-\left(\frac{K}{2}\boldsymbol{\phi}^{\mathsf{T}}(C^{-1}+\frac{2\eta}{K}C^{-1}AC^{-1})\boldsymbol{\phi} - \sum_{\boldsymbol{x}}\log\cosh(K\phi_{\boldsymbol{x}})\right)} \Big|_{\eta\to 0} \\ &= Z_{\text{HS}}^{-1} \int D\boldsymbol{\phi}\, e^{-\left(\frac{K}{2}\boldsymbol{\phi}^{\mathsf{T}}C^{-1}\boldsymbol{\phi} - \sum_{\boldsymbol{x}}\log\cosh(K\phi_{\boldsymbol{x}})\right)} \boldsymbol{\phi}^{\mathsf{T}}C^{-1}AC^{-1}\boldsymbol{\phi} \\ &= \langle \boldsymbol{\phi}^{\mathsf{T}}C^{-1}AC^{-1}\boldsymbol{\phi}\rangle_{\text{HS}} \end{aligned}$$

This means $\boldsymbol{S}$ corresponds to $C^{-1}\boldsymbol{\phi}$. In other words, $\phi_{\boldsymbol{x}}$ represents $\tilde{S}_{\boldsymbol{x}} \equiv \sum_{\boldsymbol{x}'} C_{\boldsymbol{x}\boldsymbol{x}'} S_{\boldsymbol{x}'}$, i.e., a weighted sum of spins in a local cluster. (Thus, we recover the original simple and intuitive derivation of ϕ^4 theory.)

Exercise 3.1: Consider an Ising model with only 4 spins.

$$\mathcal{H} = -K(S_1S_2 + S_3S_4) - K'(S_1S_3 + S_2S_4 + S_1S_4 + S_2S_3)$$

By coarse-graining $\phi_1 \equiv \frac{1}{2}(S_1 + S_2)$ and $\phi_2 \equiv \frac{1}{2}(S_3 + S_4)$, obtain the **exact** effective Hamiltonian in terms of ϕ_1 and ϕ_2, and verify the existence of terms proportional to ϕ^2, ϕ^4 and $|\nabla\phi|^2(= (\phi_1 - \phi_2)^2)$, respectively. (If necessary, solve numerically by setting some numerical values of your choice to K and K'.)