We will apply the GBF variational principles to the BCS Hamiltonian. ($\mathcal{H} = \mathcal{H}_{BCS}$)

For the trial Hamiltonian we take a genral quadratic Hamiltonian

$$\mathcal{H}_0 = \sum_k E_k (\alpha_k^+ \alpha_k + \beta_k^+ \beta_k) \quad (E_k \geq 0)$$

where α_k and β_k are related to $c_{k\uparrow}$ and $c_{-k\downarrow}$ by

$$\begin{pmatrix} c_{k\uparrow} \\ c^+_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \begin{pmatrix} \alpha_k \\ \beta_k^+ \end{pmatrix} \quad \begin{pmatrix} u = \cos\theta_k \\ v = \sin\theta_k \end{pmatrix}$$

with θ_k being variational parameters in addition to E_k. Then, the variational free energy is

$$F_v = \langle \mathcal{H}_{BCS} \rangle_0 - S_0 T$$

$$\langle \mathcal{H}_{BCS} \rangle_0 = \langle K \rangle_0 + \langle V \rangle_0$$

$$\langle K \rangle_0 = \sum_{k\sigma} \xi_k \langle c^+_{k\sigma} c_{k\sigma} \rangle \quad (\xi_k \equiv \epsilon_k - \mu)$$

$$= \sum_k \xi_k \langle c^+_{k\uparrow} c_{k\uparrow} + c^+_{-k\downarrow} c_{-k\downarrow} \rangle_0$$

$$= \sum_k \xi_k \langle (u_k \alpha_k^+ + v_k \beta_k)(u\alpha + v\beta^+) + (-v\alpha + u\beta^+)(-v\alpha^+ + u\beta) \rangle_0$$

$$= \sum_k \xi_k \langle (u^2 - v^2)(\alpha^+\alpha + \beta^+\beta) + 2v^2 \rangle_0 \quad \left(f_k \equiv \langle \alpha_k^+ \alpha_k \rangle = \frac{1}{e^{\beta E_k} + 1} \right)$$

$$= \sum_k \xi_k \left((\cos 2\theta_k)(2 f_k) + (1 - \cos 2\theta_k) \right)$$

$$= \sum_k \xi_k \left((\cos 2\theta_k)(2 f_k - 1) + 1 \right)$$

$$\langle V \rangle_0 = -\frac{g}{\Lambda} \sum_{kk'}{}' \langle c^\dagger_{k'\uparrow} c^\dagger_{-k'\downarrow} c_{-k\downarrow} c_{k\uparrow} \rangle_0$$

$$\langle c^\dagger_{k'\uparrow} c^\dagger_{-k'\downarrow} c_{-k\downarrow} c_{k\uparrow} \rangle_0$$

$$= \langle c^\dagger_{k'\uparrow} c^\dagger_{-k'\downarrow} \rangle_0 \langle c_{-k\downarrow} c_{k\uparrow} \rangle_0 - \cancel{\langle c^\dagger_{k'\uparrow} c_{-k\downarrow} \rangle_0 \langle c^\dagger_{-k'\downarrow} c_{k\uparrow} \rangle_0}$$

$$+ \cancel{\langle c^\dagger_{k'\uparrow} c_{k\uparrow} \rangle_0 \langle c^\dagger_{-k'\downarrow} c_{-k\downarrow} \rangle_0}$$

($\because$ Wick's theorem)
(see, e.g., Negele Orland p.76)

$$= \langle c^\dagger_{k'\uparrow} c^\dagger_{-k'\downarrow} \rangle_0 \langle c_{-k\downarrow} c_{k\uparrow} \rangle_0$$

($\because$ The 2nd term vanishes because $c^\dagger_{k'\uparrow} c_{-k\downarrow}$ doesn't preserve the total spin. The 3rd term vanishes unless $k'=k$, which yields a term $O(N^0)$ (irrelevant).)

$$\langle c_{-k\downarrow} c_{k\uparrow} \rangle_0$$

$$= \langle (-v\alpha^\dagger + u\beta)(u\alpha + v\beta^\dagger) \rangle_0$$

$$= \langle -uv(\alpha^\dagger\alpha - \beta\beta^\dagger) + u^2\beta\alpha - v^2\alpha^\dagger\beta^\dagger \rangle_0$$

$$= -uv(2f_k - 1) = -\frac{1}{2}\sin 2\theta_k (2f_k - 1)$$

$$\therefore \quad \langle V \rangle_0 = -\frac{\Lambda}{g}\Delta^2$$

$$\Delta \equiv \frac{g}{\Lambda}\sum_k \langle c_{-k\downarrow} c_{k\uparrow} \rangle_0 = -\frac{g}{2\Lambda}\sum_k (\sin 2\theta_k)(2f_k - 1)$$

$$S_0 = -2k_B \sum_k \left(f_k \log f_k + (1-f_k)\log(1-f_k) \right) \quad \text{—— } (*)$$

(von Neumann entropy)

This 2 comes from the fact that we have 2 species (α and β).

Wick's therem (for fermionic operators)

For a system described by a Hamiltonian of the form $\mathcal{H} = \sum_i \epsilon_i a_i^\dagger a_i$ with some fermionic oprators $\{a_i\} \{a_i^\dagger\}$,
the following identity holds for $2n$ $(n \geq 1)$ arbitrary linear combinations of $a_1, a_1^\dagger, a_2, a_2^\dagger, \cdots$:

$$\langle A_1 A_2 \cdots A_{2n} \rangle = \sum_{P:\text{pairing}} (-1)^P \prod_{l=1}^{n} \langle A_{P(2l-1)} A_{P(2l)} \rangle ,$$

$\left(\sum_{P:\text{ pairing}} = \text{the sum over all permutations, s.t. } P(2l-1) < P(2l), P(2l+1) \right)$

i.e., the $2n$-operator correlation function is decomposed into the sum of pair decompositions.

Example $(n=2)$

$$\langle ABCD \rangle = \langle AB \rangle\langle CD \rangle - \langle AC \rangle\langle BD \rangle + \langle AD \rangle\langle BC \rangle \quad \text{—1}$$

<u>Proof:</u> We first prove the theorem for $n=2$.

$$\langle ABCD \rangle = \mathrm{Tr}(\rho ABCD) \qquad \rho \equiv \frac{1}{Z} e^{-\beta \mathcal{H}}$$

$$= \mathrm{Tr}(\rho(\{AB\} - BA)CD) = \mathrm{Tr}(\rho \{AB\} CD) - \mathrm{Tr}(\rho BACD)$$

$$= \mathrm{Tr}(\rho \{AB\} CD) - \mathrm{Tr}(\rho B \{AC\} D) + \mathrm{Tr}(\rho BCAD)$$

$$= \mathrm{Tr}(\rho \{AB\} DC) - \mathrm{Tr}(\rho B \{AC\} D) + \mathrm{Tr}(\rho BC \{AD\})$$
$$- \mathrm{Tr}(\rho BCDA) \qquad (\{AB\} = AB + BA)$$

$$\mathrm{Tr}(\rho BCDA) = \mathrm{Tr}(\rho A(\beta) BCD) \qquad \left(A(\beta) \equiv e^{\beta \mathcal{H}} A e^{-\beta \mathcal{H}} \right)$$

If $A = a_k$, $A(\beta) = a_k(\beta) = e^{-\beta\epsilon_k} a_k$,

Then,
$$\langle ABCD\rangle = -e^{-\beta\epsilon}\langle ABCD\rangle + \langle\{AB\}CD\rangle - \langle B\{AC\}D\rangle + \langle BC\{AD\}\rangle,$$
which yields
$$\langle ABCD\rangle = \frac{1}{1+e^{-\beta\epsilon_k}}\{\langle\{AB\}CD\rangle - \langle\{AC\}BD\rangle + \langle\{AD\}BC\rangle\}$$
$$= (1-f_k)\{\qquad \text{''} \qquad\}. \quad \text{—} 2$$

If $A = a_k$ and $B = a_k^\dagger$, then,
$$(1-f_k)\{AB\} = 1-f_k = \langle AB\rangle.$$

If $A = a_k$ and B being any one of a_i or $a_i^\dagger$ but $a_k^\dagger$,
$$(1-f_k)\{AB\} = 0 \quad (= \langle AB\rangle)$$

So, $(1-f_k)\{AB\} = \langle AB\rangle$ for B being any one of $a_1, a_1^\dagger, a_2, a_2^\dagger, \cdots$. Therefore, $(1-f_k)\{AB\} = \langle AB\rangle$ for B being any linear combination of $\{a_i\}$ and $\{a_i^\dagger\}$. The same is true for C and D. Thus we've proved the statement 1 for $A = a_k$. We can prove the statement for $A = a_k^\dagger$ in much the same way. Therefore, the equation holds for A being any one of $\{a_i\}$ and $\{a_i^\dagger\}$ and any linear combinations B, C, and D.
Now, it is obvious 1 is true also for A being any linear combination. The case $n \geq 3$ can be proven by induction using the equation like 2 which reduces $2n$-products to $(2n-2)$-products. Also, the bosonic case can be proven in a similar fashion.

(*) In general, if the system is decomposed into mutually independent components like

$$\mathcal{H} = \sum_\mu \mathcal{H}_\mu \qquad \Big(\underbrace{H = \bigotimes_\mu H_\mu}_{} \Big)$$

Hilbert space decomposition
($\mathcal{H}_\mu$ acts only on H_μ)

The entropy becomes

$$S = \sum_\mu S_\mu \qquad S_\mu = \left(\begin{array}{l} \text{the entropy of the} \\ \text{subsystem } \mathcal{H}_\mu \text{ in } H_\mu \end{array} \right)$$

In the present case, $\mathcal{H} = \sum_k E_k (\alpha_k^\dagger \alpha_k + \beta_k^\dagger \beta_k)$.

So, for S_k, we consider the entropy of the subsystem

$$\mathcal{H}_{\alpha_k} \equiv E_k \alpha_k^\dagger \alpha_k$$

for which the density operator is

$$\rho_{\alpha_k} = e^{-\beta \mathcal{H}_{\alpha_k}} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\beta E_k} \end{pmatrix} \begin{array}{l} \leftarrow \alpha^\dagger \alpha = 0 \\ \leftarrow \alpha^\dagger \alpha = 1 \end{array}$$

Since this is diagonal, we can view this as a classical system. Generally, the entropy of a classical system is

$$S = k_B \sum_i (-p_i \log p_i)$$

where p_i is the probability of the state i. In the α_k system, $p_0 = 1/(1+e^{-\beta E_k}) = 1 - f_k$ $\quad p_1 = e^{-\beta E_k}/(1+e^{-\beta E_k}) = f_k$.
Therefore

$$S_{\alpha_k} = -k_B \left(f_k \log f_k + (1-f_k) \log (1-f_k) \right)$$

$$F_v = \sum_k \xi_k \left((\cos 2\theta_k)(2f_k - 1) + 1 \right)$$
$$- \frac{1}{g}\Delta^2 + 2k_B T \sum_k \left(f_k \log f_k + (1-f_k)\log(1-f_k) \right)$$

$$\Delta = -\frac{g}{2\Lambda} \sum_k (\sin 2\theta_k)(2f_k - 1)$$

Now, we consider the stationary condition with respect to the variational parameters (E_k and θ_k).

$$\frac{\partial F_v}{\partial E_k} = 0 \quad \longleftrightarrow \quad \frac{\partial F_v}{\partial f_k} = 0$$

$$0 = \frac{\partial F_v}{\partial f_k} = 2\xi_k(\cos 2\theta_k) - \frac{2\Lambda}{g}\Delta\left(-\frac{g}{\Lambda}\right)(\sin 2\theta_k)$$
$$+ 2k_B T \log(f_k/(1-f_k))$$

$$= 2\left(\xi_k \cos 2\theta_k + \Delta \sin 2\theta_k\right) - 2E_k$$

$$\longrightarrow \quad E_k = \xi_k \cos 2\theta_k + \Delta \sin 2\theta_k \quad \text{—— (1)}$$

$$0 = \frac{\partial F_v}{\partial \theta_k} = -2\xi_k(\sin 2\theta_k)(2f_k - 1)$$
$$- \frac{2\Lambda\Delta}{g}\left(-\frac{g}{2\Lambda}\right)(2\cos 2\theta_k)(2f_k - 1)$$

$$\longrightarrow \quad \xi_k \sin 2\theta_k = \Delta \cos 2\theta_k$$

$$\longrightarrow \quad \tan 2\theta_k = \frac{\Delta}{\xi_k} \quad \text{—— (2)}$$

$$\cos 2\theta_k = \frac{\pm \xi_k}{\sqrt{\xi_k^2 + \Delta^2}} \qquad \sin 2\theta_k = \frac{\pm \Delta}{\sqrt{\xi_k^2 + \Delta^2}}$$

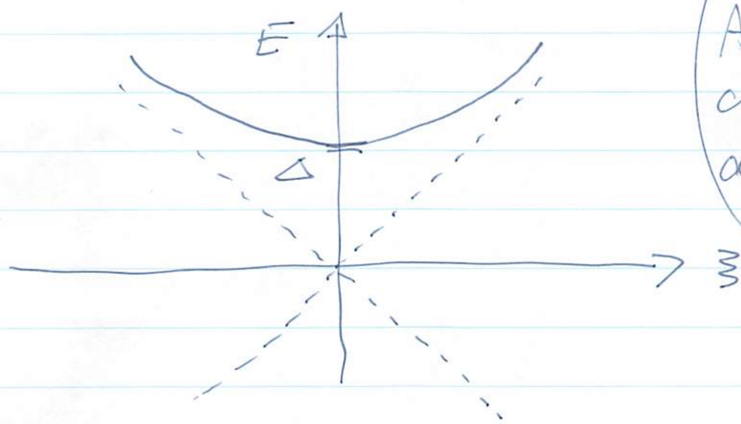
$$E_k = \pm\sqrt{\xi^2 + \Delta^2}$$

However, to make E_k positive (i.e. to make the ground state the α, β-vacuum), we take "+". So,

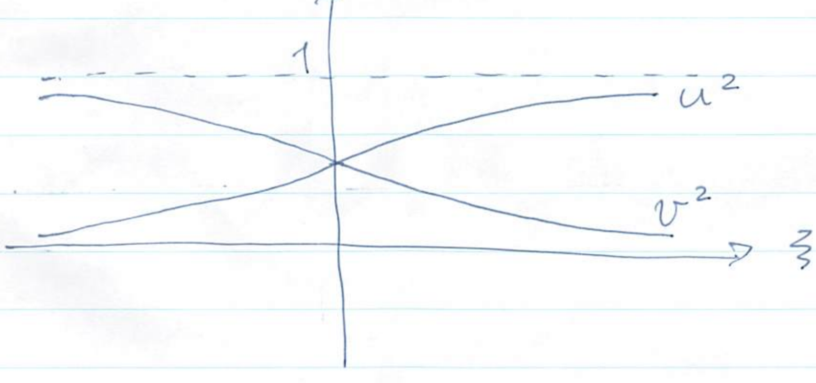
$$E_k = \sqrt{\xi^2 + \Delta^2}$$

$$u_k^2 = \frac{1}{2}(1 + \cos 2\theta) = \frac{1}{2}\left(1 + \frac{\xi}{\sqrt{\xi^2 + \Delta^2}}\right)$$

$$v_k^2 = \frac{1}{2}\left(1 - \frac{\xi}{\sqrt{\xi^2 + \Delta^2}}\right)$$



Adding the el.-ph. coupling opens up a finite excitation gap.



Self-consistent equation

$$\Delta \equiv \frac{g}{\Lambda} \sum_k{}' \langle c_{-k\downarrow} c_{k\uparrow} \rangle_0 \qquad \sum_k{}' \cdots \left(\begin{array}{l}\text{summation over}\\ \text{the shell.}\end{array}\right)$$

$$= \frac{g}{2\Lambda} \sum_k (\sin 2\theta_k)(1-2f_k)$$

$$1-2f_k = 1 - \frac{1}{e^{\beta E}+1} = \frac{e^{\beta E}-1}{e^{\beta E}+1} = \mathrm{th}\,\frac{\beta E}{2}$$

$$\Delta = \frac{g}{2\Lambda} \sum_k (\sin 2\theta_k)\, \mathrm{th}\,\frac{\beta E_k}{2}$$

$$= \frac{g}{2\Lambda} \sum_k \frac{\Delta}{E_k}\, \mathrm{th}\,\frac{\beta E_k}{2}$$

$$= \frac{g}{2} a^d \int' \frac{d\mathbf{k}}{(2\pi)^3} \frac{\Delta}{E_k}\, \mathrm{th}\,\frac{\beta E_k}{2} \qquad \left(\frac{1}{L^d} \sum_k{}' = \int' \frac{d\mathbf{k}}{(2\pi)^3} \right)$$

$$= \frac{g}{2} a^d \int' d\xi\, \rho(\xi) \frac{\Delta}{E(\xi)}\, \mathrm{th}\,\frac{\beta E(\xi)}{2}$$

$$= \frac{g}{2} \int' d\xi\, D(\xi) \frac{\Delta}{E(\xi)}\, \mathrm{th}\,\frac{\beta E(\xi)}{2} \qquad \left(\begin{array}{l} D \equiv a^d \rho \\ = \text{the DOS per} \\ \text{unit cell} \end{array}\right)$$

$$\longrightarrow \boxed{1 = g \int_{-\hbar\omega_D}^{\hbar\omega_D} d\xi\, \frac{D(\xi)}{2E(\xi)}\, \mathrm{th}\,\frac{\beta E(\xi)}{2}}$$

(gap equation)

$$E = \sqrt{\xi^2 + \Delta^2}$$

Δ at $T=0$ and T_c

Since $D(\xi)$ would not change much in the interval $\xi \in [-\hbar\omega_D, \hbar\omega_D]$, we can simply replace it by $D_F \equiv D(0)$.

$$1 = g D_F \int_0^{\hbar\omega_D} d\xi \, \frac{1}{E} \, \mathrm{th} \frac{\beta E}{2} \quad \text{—— (1)}$$

- Introducing $x \equiv \xi/\Delta$,

$$(g D_F)^{-1} = \int_0^{\hbar\omega_D/\Delta} dx \, \frac{1}{\sqrt{1+x^2}} \, \mathrm{th}\left(\frac{\beta\Delta}{2}\sqrt{1+x^2}\right)$$

Sending $\beta \to \infty$,

$$(g D_F)^{-1} = \int_0^{\hbar\omega_D/\Delta} \frac{dx}{\sqrt{1+x^2}} = \mathrm{ash} \frac{\hbar\omega_D}{\Delta}$$

$$\frac{\hbar\omega_D}{\Delta} = \mathrm{sh} \frac{1}{g D_F} \approx \frac{1}{2} e^{1/gD_F} \quad \left(e^{\frac{2}{gD_F}} \gg 1 \right)$$

$\leftrightarrow$ (weak el-ph coupling)

$$\boxed{\Delta \approx 2\hbar\omega_D \, e^{-\frac{1}{gD_F}}} \quad (\ll \hbar\omega_D) \text{ is assumed.}$$

- The critical temperature is computed as the temperature at which $\Delta = 0$. Therefore by setting $\Delta = 0$ in (1)

$$1 = g D_F \int_0^{\hbar\omega_D} d\xi \, \frac{1}{\xi} \, \mathrm{th} \frac{\beta_c \xi}{2}$$

By introducing $x \equiv \frac{\beta\xi}{2}$,

$$(g D_F)^{-1} = \int_0^{\frac{\beta_c \hbar\omega_D}{2}} dx \, \frac{1}{x} \, \mathrm{th}\, x$$

Generally, for $X \gg 1$ (i.e. $k_B T_c \ll \hbar\omega_D$)

$$\int_0^X dx \frac{\mathrm{th}\,x}{x} = \left[\mathrm{th}\,x \log x\right]_0^X - \int_0^X dx \frac{\log x}{\mathrm{ch}^2 x}$$

$$\simeq \log X - \int_0^\infty dx \frac{\log x}{\mathrm{ch}^2 x}$$

$$= \log X + \log \frac{4e^\gamma}{\pi} \quad \left(\begin{array}{l}\gamma : \text{Euler's constant} \\ \sim 0.577216\end{array}\right)$$

Therefore,

$$(g D_F)^{-1} \doteqdot \log \frac{\hbar\omega_D}{2k_B T_c} + \log \frac{4e^\gamma}{\pi} \quad \left(\frac{\hbar\omega_D}{k_B T_c} \gg 1\right)$$

$$\boxed{k_B T_c = \frac{2e^\gamma}{\pi}\hbar\omega_D e^{-\frac{1}{gD_F}}} \sim 1.13\,\hbar\omega_D e^{-\frac{1}{gD_F}} \ll \hbar\omega_D$$

Therefore,

$$\frac{2\Delta(0)}{k_B T_c} = \frac{4\pi}{2e^\gamma} = \frac{2\pi}{e^\gamma} \doteqdot 3.528$$

The excitation gap $2\Delta(0)$ can be measured for real superconductors by photo-electroric emission experiments. To quote some of them,

$$\frac{2\Delta(0)}{k_B T_c} = 3.53 \ (\text{for Al})$$
$$3.65 \ (\quad \text{In})$$
$$3.44 \ (\quad \text{Zn})$$
$$\vdots \qquad \vdots$$

Surprizingly good for the crude approximation. One of the reasons is the ratio doesn't depend on the coupling strength, which is hard to estimate accurately.

We have seen that in the self-consistent eq.

$$(gD_F)^{-1} = \underbrace{\int_0^{\hbar\omega_D} d\xi \ \frac{1}{E} \tanh\frac{\beta E}{2}}_{\equiv I(\tau, \Delta)} \quad \left(E = \sqrt{\xi^2 + \Delta^2}\right)$$

the integral on the RHS can be expanded when $\Delta = 0$ as,

$$I = (gD_F)^{-1} - \log\frac{T}{T_c} + \mathcal{O}(\tau)$$

$$= (gD_F)^{-1} - \tau + \mathcal{O}(\tau) \qquad \left(\tau \equiv \frac{T - T_c}{T_c}\right)$$

Now, we want to know the Δ dependence of I near $\tau = 0$. Since $I(\tau, \Delta) = (gD_F)^{-1}$ at $\tau = \Delta = 0$ and $I(\tau, \Delta)$ depends on Δ only through Δ^2, probably it takes the form

$$I(\tau, \Delta) = I(0, 0) - \tau - A(\beta\Delta)^2 + \mathcal{O}(\tau^2, \Delta^4)$$

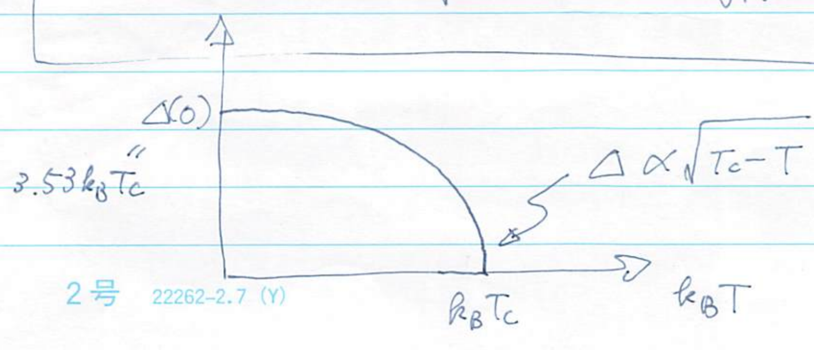
Indeed a more detailed calculation yields

$$A = \frac{7\zeta(3)}{4\pi^2} \doteqdot 0.213139 \quad \left(\text{see the next page for details}\right)$$

Putting together,

$$0 = -\tau - A(\beta\Delta)^2 + \mathcal{O}(\tau^2, (\beta\Delta)^4)$$

$$\therefore \boxed{\Delta \approx k_B T_c \sqrt{\frac{-\tau}{A}} = \frac{k_B T_c}{\sqrt{A}} \sqrt{\frac{T_c - T}{T_c}}}$$



③ $\Delta(T)$ near T_c

$$1 = gD_F \underbrace{\int_0^{\hbar\omega_D} d\xi \frac{1}{E} \tanh\frac{\beta E}{2}}_{\shortparallel \ I(\Delta)}$$

$$(gD_F)^{-1} = I(0) + (I(\Delta) - I(0))$$

From ① (9-9) $\quad I(0) = \log \frac{2e^\gamma}{\pi}\frac{\hbar\omega_D}{k_B T}$

$$= \log \frac{2e^\gamma}{\pi}\frac{\hbar\omega_D}{k_B T_c} + \log\frac{T_c}{T}$$

$$= (gD_F)^{-1} + \log\frac{T_c}{T}$$

$$I(\Delta) - I(0) = \int_0^{\hbar\omega_D} d\xi \left(\frac{1}{E}\tanh\frac{\beta E}{2} - \frac{1}{\xi}\tanh\frac{\beta\xi}{2}\right)$$

$k_B T_c, \Delta \ll \hbar\omega_D$

$$= \int_0^{\infty} d\xi \left(\frac{1}{E}\tanh\frac{\beta E}{2} - \frac{1}{\xi}\tanh\frac{\beta\xi}{2}\right)$$

$\frac{\beta\xi}{2} \equiv x$

$$= \cancel{\int_0^\infty dx \frac{\beta^2\Delta^2}{4}\frac{1}{x^3}\left(\frac{1}{\cosh^2 x} - \frac{1}{x}\tanh x\right)}$$

$$= \cancel{\left(\frac{\beta\Delta}{2}\right)^2 \times I_1}$$

$$\cancel{I \equiv \int_0^\infty dx \frac{1}{x^3}\left(\frac{1}{\cosh^2 x} - \frac{1}{x}\tanh x\right)}$$

$$\int_0^\infty d\xi \left(\frac{1}{E}\tanh\frac{\beta E}{2} - \frac{1}{\xi}\tanh\frac{\beta\xi}{2}\right)$$

$$\fallingdotseq \int_0^\infty d\xi \quad \Delta \frac{dE}{d\Delta}\left(\frac{d}{d\xi}\left(\frac{1}{\xi}\tanh\frac{\beta\xi}{2}\right)\right)$$

$$= \int_0^\infty d\xi \quad \Delta \frac{2\Delta}{2\xi} \frac{d}{d\xi}\left(\frac{1}{\xi}\tanh\frac{\beta\xi}{2}\right)$$

$$= \Delta^2 \int_0^\infty d\xi \quad \frac{1}{\xi}\frac{d}{d\xi}\left(\frac{1}{\xi}\tanh\frac{\beta\xi}{2}\right)$$

$$x \equiv \beta\xi \quad (\xi = k_B T x)$$

$$= \left(\frac{1}{\beta}\right)^{-2}\Delta^2 \int_0^\infty dx \quad \frac{1}{x}\frac{d}{dx}\left(\frac{1}{x}\tanh\frac{x}{2}\right)$$

$$= (\beta\Delta)^2 \times (-A)$$

$$A \equiv -\int_0^\infty dx \quad \frac{1}{x}\frac{d}{dx}\left(\frac{1}{x}\tanh\frac{x}{2}\right) = \frac{7\zeta(3)}{4\pi^2} \fallingdotseq 0.213139$$

(see Shiba p.141)

putting together,

$$0 = \log\frac{T_c}{T} - A(\beta_c\Delta)^2 \qquad T = T_c - \delta$$

$$\beta_c\Delta = \frac{1}{\sqrt{A}}\left(\log\frac{T_c}{T}\right)^{1/2}$$

$$\Delta = \frac{k_B T_c}{\sqrt{A}}\left(-\log\left(1-\frac{\delta}{T_c}\right)\right)^{1/2} = \frac{k_B T_c}{\sqrt{A}}\left(\frac{\delta}{T_c}\right)^{1/2}$$

$$\boxed{\Delta \fallingdotseq \frac{k_B T_c}{\sqrt{A}}\sqrt{\frac{T_c - T}{T_c}}} = \sqrt{\frac{4}{7\zeta(3)}}\,\pi\, k_B T_c \sqrt{\frac{T_c - T}{T_c}}$$

Specific Heat

$$S = -2k_B \sum_k \left(f_k \log f_k + (1-f_k)\log(1-f_k) \right)$$

$$C = T\frac{dS}{dT}$$

$$= -2k_B T \sum_k \frac{df_k}{dT}\left(\log f_k - \log(1-f_k) \right)$$

$$= -2k_B T \sum_k \frac{d(\beta E_k)}{dT}\frac{df_k}{d(\beta E_k)} \log\frac{f_k}{1-f_k}$$

$$= -2k_B T \sum_k \left(-\frac{E}{k_B T^2} + \beta\frac{1}{2E}\frac{d\Delta^2}{dT} \right)\left(\frac{1}{\beta}\frac{df}{dE} \right)\log e^{-\beta E}$$

$$= -2k_B T \sum_k \left(-\frac{E}{k_B T^2} + \frac{1}{2k_B T E}\frac{d\Delta^2}{dT} \right)\left(\frac{1}{\beta}\frac{df}{dE} \right)(-\beta E)$$

$$= 2k_B T \frac{1}{k_B T^2} \sum_k \left(E^2 - \frac{T}{2}\frac{d\Delta^2}{dT} \right)\left(-\frac{df}{dE} \right)$$

$$= \frac{2}{T} \sum_k \left(E^2 - \frac{T}{2}\frac{d\Delta^2}{dT} \right)\left(-\frac{df}{dE} \right)$$

$$\approx \frac{2}{k_B T^2} N \int d\xi\, D(\xi) \left(E^2 - \frac{T}{2}\frac{d\Delta^2}{dT} \right)\frac{e^{\beta E}}{(e^{\beta E}+1)^2}$$

$$C/N \approx \frac{2}{k_B T^2} D_F \int d\xi \left(E^2 - \frac{T}{2}\frac{d\Delta^2}{dT} \right)\frac{e^{\beta E}}{(e^{\beta E}+1)^2} \quad \text{——} (1)$$

At low temperature, $\left(E = \Delta + \frac{\xi^2}{2\Delta} \right)$

$$C/N \sim \frac{2}{T} D_F \int d\xi \; E^2 \frac{e^{\beta E}}{(e^{\beta E}+1)^2}$$

$$\sim \frac{2}{T} D_F \int d\xi \; \Delta^2 e^{-\beta\Delta - \frac{\beta\xi^2}{2\Delta}}$$

$$\sim \frac{2}{k_B T^2} D_F \Delta^2 e^{-\beta\Delta} \sqrt{\frac{2\pi\Delta}{\beta}}$$

$$= \frac{2 k_B}{(k_B T)^{3/2}} D_F \Delta^{\frac{5}{2}} \sqrt{2\pi} \; e^{-\beta\Delta}$$

$$\boxed{\frac{C}{N} = k_B \sqrt{2\pi} \frac{D_F \Delta^{5/2}}{(k_B T)^{3/2}} e^{-\beta\Delta}}$$

Near the critical temperature.

$\Delta \sim 0 \quad E \sim \xi$

(1) The contribution from the $\underbrace{E^2 \sim \xi^2 \text{ term}}_{\text{in 9-11-①}}$ is the same as the specific heat of the normal (Fermi sea) state:

$$C_n/N \sim \frac{2}{k_B T^2} D_F \int_{-\infty}^{\infty} d\xi \, \xi^2 \frac{e^{\beta\xi}}{(e^{\beta\xi}+1)^2}$$

$$= 2 k_B (k_B T) D_F \int_{-\infty}^{\infty} dx \frac{x^2 e^x}{(e^x+1)^2}$$

$$= 2B\, D_F\, k_B (k_B T)$$

$$B \equiv \int_{-\infty}^{\infty} dx \frac{x^2 e^x}{(e^x+1)^2} = \frac{2\pi^2}{3}$$

$$C_n/N k_B = \frac{4\pi^2}{3} D_F\, k_B T \approx 13.159\, D_F\, k_B T$$

(2) The contribution from the $-\frac{T}{2}\frac{d\Delta^2}{dT}$ term is (just below T_c)

$$C_\Delta/N k_B = \beta_c^2\, 2 D_F \left(-\frac{T_c}{2}\right) \int_{-\infty}^{\infty} d\xi \frac{(k_B T_c)^2}{A}\left(-\frac{1}{T_c}\right) \frac{e^{\beta\xi}}{(e^{\beta\xi}+1)^2}$$

$$= \frac{D_F}{A} k_B T_c \int_{-\infty}^{\infty} dx \frac{e^x}{(e^x+1)^2} \qquad \left(A = \frac{7\zeta(3)}{4\pi^2}\right)$$

$\because$ 9-10-③

$$= A^{-1} D_F\, k_B T_c \qquad \left(\Delta^2 \approx \frac{k_B^2 T_c}{A}(T_c - T)\right)$$

$\parallel$

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$$\int_{-\infty}^{\infty} dx \, \frac{x^2 e^x}{(e^x+1)^2} =$$

$$= 2\int_0^{\infty} dx \; x^2 \left(\frac{-1}{e^x+1} \right)'$$

$$= 2\left[\frac{-x^2}{e^x+1} \right]_0^{\infty} + 2\int_0^{\infty} dx \; 2x \, \frac{1}{e^x+1}$$

$$= 4 \int_0^{\infty} dx \, \frac{x}{e^x+1} = 4 I_1$$

$$I_n \equiv \int_0^{\infty} dx \, \frac{x^n}{e^x+1} = \int_0^{\infty} dx \; x^n e^{-x} \sum_{k=0}^{\infty} e^{-kx}$$

$$= \sum_{k=0}^{\infty} \int_0^{\infty} dx \; e^{-(k+1)x} x^n$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k+1)^{n+1}} \int_0^{\infty} dx \; e^{x} x^n$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k+1)^{n+1}} \times \Gamma(n+1)$$

$$= \zeta(n+1) \times \Gamma(n+1)$$

$$\therefore \; B = 4 \, \zeta(2) \, \Gamma(2)$$

$$= 4 \cdot \frac{\pi^2}{6} \cdot 1 = \frac{2\pi^2}{3}$$

$$C = C_{el} + C_{ph}$$

$$C_{el} = \begin{cases} (\text{const}) \times \dfrac{1}{T^{3/2}} e^{-\beta\Delta(0)} & (T \sim 0) \\ C_n + \Delta C & (T \sim T_c) \end{cases}$$

$$C_n = 13.2\, N D_F k_B^2 T \qquad (T \sim T_c)$$

$$\Delta C \approx 4.7\, N D_F k_B^2 T_c \qquad (T \sim T_c)$$

$$C_{ph} \approx \lambda T^3$$