Spin-Wave Theory of Antiferromagnets

The Ground State

- The ground state of the antiferromagnetic Heisenberg model

The antiferromagnetic case is more complicated. Our intuition and also mean-field approximations suggest that the ground state is a state in which the spins alternate, e.g., something like checkerboard in the case of the square lattice:

$$\begin{array}{cccc} \uparrow & \downarrow & \uparrow & \downarrow \\ \downarrow & \uparrow & \downarrow & \uparrow \\ \uparrow & \downarrow & \uparrow & \downarrow \end{array} \quad \cdots \text{ Neél State}$$

However, it's also easy to show that the perfect Neél state defined by

$$S_i^z |\Psi\rangle = \begin{cases} +\frac{1}{2}|\Psi\rangle & (i \in A\text{-sublattice}) \\ -\frac{1}{2}|\Psi\rangle & (i \in B\text{-sublattice}) \end{cases}$$

can't be the ground state. (Show it!)
Instead, the true ground state is "imperfect" Neél state:

$$\langle\Psi|S_i^z|\Psi\rangle = \begin{cases} +m & (i \in A) \\ -m & (i \in B) \end{cases}$$

with $|m| < \frac{1}{2}$, even if the Neél state is the ground state in someway. In other words, we have some quantum fluctuation even at $T=0$ in the antiferromagnetic model. In what follows, we estimate the amplitude of the quantum fluctuation and discuss the Neél state is really the ground state or not.

※ The simplest mean-field approximation to the Heisenberg model yields exactly the same result as the mean-field approximation to the Ising model. Therefore, it predicts finite temperature phase transitions for all spatial dimensions, and spontaneous (staggered) magnetization for the ground state. This result is correct for $d \geq 3$ However, it is widely accepted that in $d=2$ there is no finite-T phase transition in the Heisenberg model though the ground state still has the spontaneous magnetization. In $d=1$, the true ground state is not the Neél state at all.

$$\mathcal{H} = -J \sum_{(x x')} \left(S_x^x S_{x'}^x + S_x^y S_{x'}^y + S_x^z S_{x'}^z \right)$$

$$\xrightarrow[\text{mean-field approx}]{} -J \sum_{(x x')} \left(S_x^x \langle S_{x'}^x \rangle + S_x^y \langle S_{x'}^y \rangle + S_x^z \langle S_{x'}^z \rangle + \langle S_x^x \rangle S_{x'}^x + \cdots \right)$$

$$= -zJ \sum_x \left(m^x S_x^x + m^y S_x^y + m^z S_x^z \right)$$

$$= -zJ \sum_x \mathbb{m} \cdot \mathbb{S}_x \qquad \left(\mathbb{m} \equiv \langle \mathbb{S}_x \rangle \right)$$

$$\xrightarrow[\text{basis rotation}]{} -zJm \sum S_x^z$$

(the same as the mean-field Hamiltonian for the Ising model)

Spin wave theory

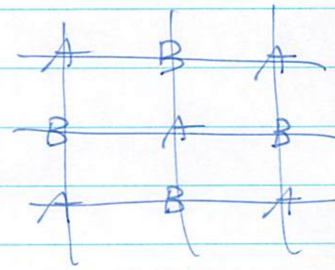
- The antiferromagnetic case can also be treated in a similar way to the ferromagnetic case.

- However, there are a few differences.

  (i) The ground state is not the fully-polarized state. This means that even in the low-T limit, the number of "flipped" spins doesn't go to zero. This causes the quantum shrink of the magnetization. Because of this, we should not expect that the approximation is asymptotically exact in the $T \to 0$ limit, unlike ferromagnets.

  (ii) The dispersion relation is essentially different: the excitation energy is proportional to k, in contrast to k^2 for the ferromagnetic case. This changes all the T-dependences of various quantities in low-T limit.

- Hamiltonian



$$\mathcal{H} = J \sum_{(x,x')} \mathbf{S}_x \cdot \mathbf{S}_{x'}$$

- The ground state

In contrast to the ferromagnetic case, we don't know any compact expression for the ground state. We just know (or at least assume) that we have finite "staggard" magnetization:

$$m \equiv \frac{1}{N} \sum_x (-1)^x S^z_x \neq 0 \qquad (-1)^x = \begin{cases} 1 & (x \in A) \\ -1 & (x \in B) \end{cases}$$

because of the spontaneous symmetry breaking. Note that the "fully-polarized" state ($|m| = S$) cannot be the exact ground state. This is in a strong contrast to the ferromagnetic case where $|m| = 1$ state is the exact ground state.

- The magnon creation/annihilation

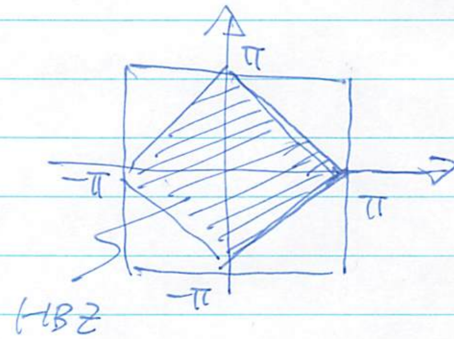
For the spins on the A sublattice ($x \in A$) we do the same as in the ferromagnetic case, and we swap the creation op. and the annihilation op. for the B sublattice:

$$\begin{array}{lcl cl} S^z_x & \Rightarrow & S - a^\dagger_x a_x & \Rightarrow & -S + b^\dagger_x b_x \\ S^+_x & \Rightarrow & \sqrt{2S}\, a_x & \Rightarrow & \sqrt{2S}\, b^\dagger_x \\ S^-_x & \Rightarrow & \sqrt{2S}\, a^\dagger_x & \Rightarrow & \sqrt{2S}\, b_x \\ & & (\text{on } A) & & (\text{on } B) \end{array}$$

∘ The Fourier decomposition with the half Brillouin zone (HBZ)

$$a_x = \frac{1}{\sqrt{\Lambda}} \sum_{k \in HBZ} e^{ikx} a_k$$

$$b_x = \frac{1}{\sqrt{\Lambda}} \sum_{k \in HBZ} e^{ikx} b_k$$



$$\Lambda \equiv N/2$$

∘ The Hamiltonian in terms of a_x and $b_{x'}$.

$$S^z_x S^z_{x'} \Rightarrow -(S - a^\dagger_x a_x)(S - b^\dagger_{x'} b_{x'})$$
$$= -S^2 + S(a^\dagger_x a_x + b^\dagger_{x'} b_{x'}) - \underbrace{a^\dagger_x a_x b^\dagger_{x'} b_{x'}}$$
$$\Rightarrow -S^2 + S(a^\dagger_x a_x + b^\dagger_{x'} b_{x'})$$

can be neglected if the excitation is sparse.

$$S^+_x S^-_{x'} + S^-_x S^+_{x'} \Rightarrow 2S(a_x b_{x'} + a^\dagger_x b^\dagger_{x'})$$

$$\therefore \quad \mathbf{S}_x \cdot \mathbf{S}_{x'} \Rightarrow -S^2 + S(a^\dagger_x a_x + b^\dagger_{x'} b_{x'} + a_x b_{x'} + a^\dagger_x b^\dagger_{x'})$$

$$\mathcal{H} \Rightarrow J \sum_{(x,x')} \left(-S^2 + S(a^\dagger_x a_x + b^\dagger_{x'} b_{x'} + a_x b_{x'} + a^\dagger_x b^\dagger_{x'})\right)$$

$$= -dNJS^2$$
$$※ \quad + (2dJS) {\sum_k}' \left((a^\dagger_k a_k + b^\dagger_{\bar{k}} b_{\bar{k}}) + \gamma_k (a_k b_{\bar{k}} + a^\dagger_k b^\dagger_{\bar{k}})\right)$$

$$\left({\sum_k}' \equiv \sum_{k \in HBZ} \right)$$

$$\left(\bar{k} \equiv -k \quad \gamma_k = \frac{1}{d} \sum_\delta \cos(k \cdot \delta) \approx 1 - \frac{1}{2d} k^2 a^2 \right)$$

◦ Calculation of ※

$$\sum_{(x,x)} a_x^+ a_x = 2d \sum_{x\in A} a_x^+ a_x$$

$$= 2d \frac{1}{\Lambda} \sum_{x\in A} \sum_{k,k'}' e^{-ikx+ik'x} a_k^+ a_{k'}$$

$$= 2d \sum_{k} a_k^+ a_{k'} \qquad \left(\sum_{x\in A} e^{ikx} = \Lambda \delta_{k,0} \right)$$

$$\sum_{(x'x')} b_{x'}^+ b_{x'} = 2d \sum_{x'\in B} b_{x'}^+ b_{x'}$$

$$= 2d \frac{1}{\Lambda} \sum_{x'\in B} \sum_{k,k'}' e^{-ikx'+ik'x'} b_k^+ b_{k'}$$

$$= 2d \sum_{k}' b_k^+ b_k$$

$$\left(\sum_{x'\in B} e^{ik\cdot x'} = \sum_{x\in A} e^{ik(x+\stackrel{\exists}{\delta})} = \Lambda \delta_{k,0} e^{ik\delta} = \Lambda \delta_{k,0} \right)$$

$$\sum_{(x,x')} a_x b_{x'} = \sum_{x\in A} \sum_{\delta} \frac{1}{\Lambda} \sum_{kk'} e^{ikx+ik'(x+\delta)} a_k b_{k'}$$

$$= \sum_{kk'}' \sum_{\delta} \delta_{k,-k'} e^{ik'\delta} a_k b_{k'}$$

$$= \sum_{k}' 2d\, \gamma_k a_k b_{\bar{k}}$$

$$\left(\gamma_k \equiv \frac{1}{2d} \sum_{\delta} e^{ik\delta} = \frac{1}{d} \left(\cos(k^x a) + \cos(k^y a) + \cos(k^z a) \right) \right)$$

∘ Bogoliubov transformation

$$\mathcal{H} = -dNJS^2 + \Delta\mathcal{H}$$

$$\Delta\mathcal{H} = t\sum_k{}'\left((a_k^+ a_k + b_{\bar{k}}^+ b_{\bar{k}}) + \gamma_k (a_k b_{\bar{k}} + a_k^+ b_{\bar{k}}^+)\right)$$

$$(t \equiv 2dJS)$$

$$\Delta\mathcal{H} = t\sum_k{}'\left((a_k^+ a_k + b_{\bar{k}} b_{\bar{k}}^+ - 1) + \gamma(a_k b_{\bar{k}} + a_{\bar{k}}^+ b_{\bar{k}}^+)\right)$$

$$= t\sum_k{}'\left\{(a_k^+, b_{\bar{k}})\begin{pmatrix} 1 & \gamma_k \\ \gamma_k & 1 \end{pmatrix}\begin{pmatrix} a_k \\ b_{\bar{k}}^+ \end{pmatrix} - 1\right\}$$

Here we define a new set of boson operators α_k, β_k by

$$\begin{pmatrix} a_k \\ b_{\bar{k}}^+ \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ -v_k & u_k \end{pmatrix}\begin{pmatrix} \alpha_k \\ \beta_k^+ \end{pmatrix} \qquad \begin{pmatrix} u_k^2 - v_k^2 = 1 \\ \text{Bogoliubov trans.} \end{pmatrix}$$

The condition $u^2 - v^2 = 1$ guarantees that the new ops. satisfies $[\alpha_k, \alpha_{k'}^+] = [\beta_k, \beta_{k'}^+] = \delta_{kk'}$.
The matrix elements u_k, v_k are determined so that the α and β particles are decoupled. Namely, we choose u and v so that $U\Gamma U$ is a diagonal matrix, where $\Gamma \equiv \begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix}$ and $U \equiv \begin{pmatrix} u & -v \\ -v & u \end{pmatrix}$.

Parametrizing u and v by $(u_k, v_k) = (\mathrm{ch}\,\theta_k, \mathrm{sh}\,\theta_k)$, we have

$$U^+\Gamma U = \begin{pmatrix} \mathrm{ch}2\theta - \gamma\,\mathrm{sh}2\theta & \gamma\,\mathrm{ch}2\theta - \mathrm{sh}2\theta \\ \gamma\,\mathrm{ch}2\theta - \mathrm{sh}2\theta & \mathrm{ch}2\theta - \gamma\,\mathrm{sh}2\theta \end{pmatrix}$$

Then by choosing θ s.t. $\tanh 2\theta = \gamma$,

$$U^{\dagger} \Gamma U = \sqrt{1-\gamma^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

For this choice, the Hamiltonian becomes

$$\Delta \mathcal{H} = \sum_k{}' \epsilon_k (\alpha_k^{\dagger} \alpha_k + \beta_k \beta_k^{\dagger}) - \frac{Nt}{2}$$

$$\epsilon_k \equiv t\sqrt{1-\gamma^2} \qquad t \equiv 2dJS$$

$$\gamma_k \equiv \frac{1}{d} \sum_\alpha \cos(k^\alpha a)$$

$$\simeq 1 - \frac{1}{2d} k^2 a^2$$

$$\left(\begin{array}{l} \epsilon_k \sim 2dJS \sqrt{\frac{k^2a^2}{d}} \sim \hbar v_{sw} |k| \quad \left(\begin{array}{l}\text{linear} \\ \text{dispersion!}\end{array}\right) \\ \hbar v_{sw} \equiv 2\sqrt{d}\, JSa \end{array} \right)$$

- The total Hamiltonian

$$\mathcal{H} = -dNJS(S+1) + \sum_k{}' \epsilon_k (\alpha_k^{\dagger} \alpha_k + \beta_k^{\dagger} \beta_k + 1)$$

- The ground state energy

$$\epsilon = \frac{E}{N} = -dJS(S+1) + \frac{1}{N}\sum_k{}' \epsilon_k$$

half Brillouin zone

$$= -dJS(S+1) + \frac{2dJS}{N}\sum_k{}' \sqrt{1-\gamma_k^2}$$

$$= -dJS(S+1) + \frac{dJS}{N}\sum_k \sqrt{1-\gamma_k^2}$$

(∵ For the other half of the Brillouin zone γ_k simply changes its sign.)

$$\frac{1}{N}\sum_k \sqrt{1-\gamma_k^2} = \frac{\Omega}{N}\int_{-\pi/a}^{\pi/a} \frac{d^dk}{(2\pi)^d} \sqrt{1-\left(\frac{1}{d}\sum_{\mu=1}^{d}\cos(ak_\mu)\right)^2}$$

$$= \int_{-\pi}^{\pi} \frac{d^dx}{(2\pi)^d} \sqrt{1-\left(\frac{1}{d}\sum_{\mu=1}^{d}\cos(x_\mu)\right)^2} = I$$

$$\left(I = \frac{2}{\pi}\ (d=1),\ 0.842\ (d=2),\ 0.903\ (d=3) \right)$$

$$\Rightarrow \quad \epsilon = -dJS(S+1) + dJS\,I$$

$$= -dJS^2\left(1 + \frac{1}{S}(1-I)\right)$$

$$= \begin{cases} -JS^2\left(1+\frac{0.363}{S}\right) & (d=1) \\ -2JS^2\left(1+\frac{0.158}{S}\right) & (d=2) \\ -3JS^2\left(1+\frac{0.098}{S}\right) & (d=3) \end{cases}$$

The first term represents the energy of the classical problem whereas the second term the quantum correction.
Note that the quantum correction is smaller for larger S, as expected.

- Zero-T sublattice magnetization
  (Shrinkage of spins)

$$m = \langle S_i^z \rangle = S - \langle n \rangle \quad (i \in A\text{-sublattice})$$

$$\langle n_i \rangle = \langle a_i^+ a_i \rangle = \frac{2}{N} \sum_{i \in A} \langle a_i^+ a_i \rangle = \frac{2}{N} \sum_k{}' \langle a_k^+ a_k \rangle$$

$$= \frac{2}{N} \sum_k{}' \langle (u\alpha^+ - v\beta)(u\alpha - v\beta^+) \rangle$$

$$= \frac{2}{N} \sum_k{}' \left(u^2 \langle \alpha^+\alpha \rangle - uv \langle \alpha^+\beta^+ \rangle - vu \langle \beta\alpha \rangle + v^2 \langle \beta\beta^+ \rangle \right)$$

At $T=0$ $\quad \langle \alpha^+\alpha \rangle = \langle \alpha^+\beta^+ \rangle = \langle \beta\alpha \rangle = 0 \quad \langle \beta\beta^+ \rangle = 1$

$$= \frac{2}{N} \sum_k{}' v_k^2$$

$$v_k^2 = sh^2\theta = \frac{1}{2}(ch 2\theta - 1)$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{1-\gamma_k^2}} - 1 \right) \quad \left(\gamma_k \equiv \frac{1}{d} \sum_{\mu=1}^{d} \cos(a k_\mu) \right)$$

$$\therefore \langle n_i \rangle = \frac{2}{N} \sum_k{}' \left(\frac{1}{2} \frac{1}{\sqrt{1-\gamma_k^2}} - \frac{1}{2} \right)$$

$$= -\frac{1}{2} + \frac{1}{N} \sum_k{}' \frac{1}{\sqrt{1-\gamma_k^2}} \qquad \frac{1}{N} \sum_k{}' ch 2\theta =$$

$$= -\frac{1}{2} + \int' \frac{d^d k}{(2\pi)^d} \frac{1}{\sqrt{1-\gamma_k^2}}$$

$$= -\frac{1}{2} + \frac{1}{2} \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{1}{\sqrt{1-\gamma_k^2}}$$

$$m = S + \frac{1}{2} - \frac{1}{2} I_d$$

$$m = S + \frac{1}{2} - \frac{1}{2} I_d$$

$$I_d \equiv \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{1}{\sqrt{1-\gamma_k^2}}$$

$$I_1 = \infty \qquad I_2 \fallingdotseq 1.39 \qquad I_3 \fallingdotseq 1.16$$

$$m/S = \begin{cases} -\infty & (d=1) \\ 1 - 0.197/S & (d=2) \\ 1 - 0.078/S & (d=3) \end{cases}$$

- The reason why the spin-wave approximation yields the unphysical result in 1D is that the basic assumption is badly violated: the fluctuation from the rigid order is not small. In fact, the one-dimensional antiferromagnetic Heisenberg model does not have the long-range order to begin with in contrast to 2-d or higher, which has been established by exact solutions.

Comparison to quantum Monte Carlo

(Sandvik PRB56, 11678 (1997))

- $S = 1/2$, $d = 2$

$$\epsilon_{SW} = -\frac{J}{2}(1 + 2\times 0.158)$$

$$= -0.658\,J$$

$$e_{QMC} = -0.669437(5)\,J$$

$$m_{SW} = 0.303$$

Only 1 or 2 % error.

$$m_{QMC} = 0.3070(3)$$

.... $S = 1/2$, $d = 2$ is the case in which the quantum effect is the most significant among the cases where the spin wave approximation is self-consistent (i.e., no divergence).
The above comparison shows that the spin wave approximation is rather accurate when it is self-consistent.

- Heat capacity (the same as acoustic phonons)

$$E/\Omega = \frac{1}{\Omega}\sum_{\mathbf{k}}{}' \frac{2\epsilon_{\mathbf{k}}}{e^{\beta\epsilon_{\mathbf{k}}}-1}$$ (The constant term has been omitted.) ("2" comes from α and β.)

$$= \int' \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{2\hbar v k}{e^{\beta\hbar v k}-1}$$ $(v \equiv v_{sw})$

$\left(L^d \int_0^{k_D} \frac{d^d\mathbf{k}}{(2\pi)^d} = (\#\text{ of DOF}) = \frac{N}{2}\right)$

$$= \int^{k_D} dk \frac{4\pi k^2}{(2\pi)^3} \frac{2\hbar v k}{e^{\beta\hbar v k}-1}$$ $\left(k_D \sim \frac{\pi}{a}\right)$

$$= \frac{\hbar v}{\pi^2}\left(\frac{k_BT}{\hbar v}\right)^4 \int_0^{\frac{\hbar v k_D}{k_B T}} dx \frac{x^3}{e^x-1}$$

$$\underset{T \ll \Theta_N}{\Longrightarrow} \frac{A}{\pi^2}\frac{(k_BT)^4}{(\hbar v)^3} = \frac{\pi^2}{15}\frac{(k_BT)^4}{(\hbar v)^3}$$

$$\left(A \equiv \int_0^\infty dx \frac{x^3}{e^x-1} = \frac{\pi^4}{15}\right)$$

$$\Theta_N \equiv \hbar v k_D / k_B$$ $\left(\hbar v = \frac{k_B \Theta_N}{k_D}\right)$

= (the "Debye temperature" for magnons)

$$\epsilon \equiv E/N = \frac{\pi^2}{15}\frac{1}{n}\frac{(k_BT)^4}{(k_B\Theta_N)^3} k_D^3$$ $(k_D^3 = 3\pi^2 n)$

$$\boxed{\epsilon = \frac{\pi^4}{5} k_B \Theta_N \left(\frac{T}{\Theta_N}\right)^4 \quad (T \ll \Theta_N)}$$

$$\boxed{C = \frac{4\pi^4}{5} k_B \left(\frac{T}{\Theta_N}\right)^3 \quad (T \ll \Theta_N)}$$

∘ $A_n \equiv \int_0^\infty dx \, \frac{x^n}{e^x - 1}$

$= \int_0^\infty dx \; x^n e^{-x} \sum_{p=0}^{\infty} e^{-px}$

$= \sum_{p=0}^{\infty} \int_0^\infty dx \; e^{-(1+p)x} \, x^n$

$= \sum_{p=0}^{\infty} \frac{1}{(1+p)^{n+1}} \int_0^\infty d\xi \; e^{-\xi} \, \xi^n$

$= \zeta(n+1) \, \Gamma(n+1)$

$A_3 = \zeta(4) \, \Gamma(4) = \frac{\pi^4}{90} \times 6 = \frac{\pi^4}{15}$

∘ The value of k_D

$\frac{N}{2} = \sum_{k}{}' = \int' \frac{d^3 k}{\left(\frac{2\pi}{L}\right)^3} = \frac{\Omega}{(2\pi)^3} \int' d^3 k$

$\Rightarrow \frac{\Omega}{(2\pi)^3} \frac{4\pi}{3} k_D^3$

the definition of k_D

$\rightarrow \; k_D = \left(3\pi^2 \, N/\Omega\right)^{1/3} = \frac{(3\pi^2)^{1/3}}{a} \quad (\Omega = N a^3)$

$k_D^3 = 3\pi^2 n$

Why do spin operators commute ?

Question

In the disscussion of the spin wave approximation, we took the commutation relation

$$[S_i^\alpha, S_j^\beta] = 0 \quad (i \neq j)$$

for granted ? But what justified it ?

Answer A quick answer is that the spin ops. are derived from the fermionic ops. for electrons, but it doesn't change the number of electrons. This means that the spin ops. are product of equal number of creation and annihilation ops. As a result, swapping S_i^α and S_j^β is swapping even-number products of fermion ops. Hence the commutation.

More specifically, we can go back to the toy example of the spin exchange :

$$\mathcal{H} = K + V \quad \left(\begin{array}{l} K \equiv t \sum_\sigma \left(c_{1\sigma}^+ c_{2\sigma} + h.c. \right) \\ V \equiv \frac{u}{2} \sum_i n_i (n_i - 1) \end{array} \right)$$

The effective Hamitonian of the spin degrees of freedom can be derived by the 2-nd order pertubation. Namely, in the limit $t=0$, the ground states are 4-fold degenerated $(1\uparrow, 2\uparrow)$ $(1\downarrow, 2\downarrow)$ $(1\uparrow\ 2\downarrow)$ $(1\downarrow\ 2\uparrow)$. In this 4-dimensional reduced Hilbert sp. the Hamitonian acts as an effective Hamiltonian

$$\mathcal{H}_{eff} = -PKQ(\mathcal{H}-E_0)^{-1}QKP \qquad \text{——} ①$$

where P is the projector onto the 4-dim ground state space and $Q \equiv 1-P$, E_0 is the non-perturbative ground state energy.

Because the op. Q demands the excited state, $\mathcal{H}-E_0$ in ① can be replaced by uI:

$$\mathcal{H}_{eff} = -\frac{1}{u} PKQKP.$$

In

$$PKQKP = t^2 P \sum_{\sigma\tau} (c_{1\sigma}^{+} c_{2\sigma} + c_{2\sigma}^{+} c_{1\sigma}) Q (c_{1\tau}^{+} c_{2\tau} + c_{2\tau}^{+} c_{1\tau}) P$$

the condition imposed by P & Q makes some of the terms irrelevant. For $\sigma = \tau$, the only terms that would yield non-zero contribution are

$$t^2 \sum_{\sigma} (c_{1\sigma}^{+} c_{2\sigma} c_{2\sigma}^{+} c_{1\sigma} + c_{2\sigma}^{+} c_{1\sigma} c_{1\sigma}^{+} c_{2\sigma})$$

$$= t^2 \sum_{\sigma} (-c_{1\sigma}^{+} c_{1\sigma} c_{2\sigma}^{+} c_{2\sigma} - c_{1\sigma}^{+} c_{1\sigma} c_{2\sigma}^{+} c_{2\sigma} + c_{1\sigma}^{+} c_{1\sigma} + c_{2\sigma}^{+} c_{2\sigma})$$

$$= -2t^2 (c_{1\uparrow}^{+} c_{1\uparrow} c_{2\uparrow}^{+} c_{2\uparrow} + c_{1\downarrow}^{+} c_{1\downarrow} c_{2\downarrow}^{+} c_{2\downarrow}) + 2t^2$$

$$= -t^2 \left(\mathbb{c}_1^{+}(1+\sigma_z)\mathbb{c}_1 \, \mathbb{c}_2^{+}(1+\sigma_z)\mathbb{c}_2 + \mathbb{c}_1^{+}(1-\sigma_z)\mathbb{c}_1 \, \mathbb{c}_2^{+}(1-\sigma_z)\mathbb{c}_2 \right) + 2t^2$$

$$= -t^2 \left(2 + 4 S_1^z S_2^z \right) + 2t^2$$

$$= -4t^2 S_1^z S_2^z$$

$$\left(S_i^\alpha \equiv \frac{1}{2} \mathbb{c}_i^\dagger \sigma_\alpha \mathbb{c}_i , \quad \mathbb{c}_i \equiv \begin{pmatrix} c_{i\uparrow} \\ c_{i\downarrow} \end{pmatrix} \quad \sigma_\alpha : \text{Pauli matrix} \right)$$

The terms with $\tau = \bar{\sigma} = -\sigma$ are

$$t^2 \sum_\sigma \left(c_{1\sigma}^\dagger c_{2\sigma} c_{2\bar{\sigma}}^\dagger c_{1\bar{\sigma}} + c_{2\sigma}^\dagger c_{1\sigma} c_{1\bar{\sigma}}^\dagger c_{2\bar{\sigma}} \right)$$

$$= t^2 \sum_\sigma \left(-c_{1\sigma}^\dagger c_{1\bar{\sigma}} c_{2\bar{\sigma}}^\dagger c_{2\sigma} - c_{2\sigma}^\dagger c_{2\bar{\sigma}} c_{1\bar{\sigma}}^\dagger c_{1\sigma} \right)$$

$$= -2t^2 \left(c_{1\uparrow}^\dagger c_{1\downarrow} c_{2\downarrow}^\dagger c_{2\uparrow} + c_{1\downarrow}^\dagger c_{1\uparrow} c_{2\uparrow}^\dagger c_{2\downarrow} \right)$$

$$= -2t^2 \left(S_1^+ S_2^- + S_1^- S_2^+ \right)$$

$$= -4t^2 \left(S_1^x S_2^x + S_1^y S_2^y \right)$$

Putting together,

$$\mathcal{H}_{\text{eff}} = \frac{4t^2}{u} \left(S_1^x S_2^x + S_1^y S_2^y + S_1^z S_2^z \right) = \frac{4t^2}{u} \mathbf{S}_1 \cdot \mathbf{S}_2 \quad //$$

From this derivation, it is obvious spins are related to fermion creation/annihilation ops. by

$$S_i^\alpha = \frac{1}{2} \mathbb{c}_i^\dagger \sigma_\alpha \mathbb{c}_i .$$

As argued already, it is now obvious that $[S_1^\alpha , S_2^\beta] = 0$. This argument is essentially the same for more general cases.