Condensed Matter Physics I

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References

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Dover, New York (1975)

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高田康民 : 超伝導 (朝倉)

加藤岳生 : 物性物理学講義 (サイエンス社)

青木秀夫 : 超伝導入門 (裳華房)

Lecture notes will be put somewhere in our group web site

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The grade will be based on a test at the end of the lecture series (i.e. the 14 th lecture, most likely on July 13.).

[0] Many-Body Problems

- Human brains can't deal with many-body problems. It requires $\mathcal{O}(e^{aN})$ resources in general. To be more precise, humans can only deal with $\mathcal{O}(N^0)$ problems where N is the number of degrees of freedom involved.

- In all cases where it seems we solve many-body problems, we simply reduce the original problem into $\mathcal{O}(N^0)$ problem in some way.

ex 1) Mean-field approximation $\cdots (S_1 \cdots S_N) \rightarrow m \equiv \langle s_i \rangle$

ex 2) exact solutions $\cdots\cdots$ available only for measure-zero subset of the problem space.

Integral eq.

(Similar to the pseudo random numbers)

ex 3) Renormalization group $\cdots\cdots$ scaling dimensions precise only in the vicinity of the fixed points

ex 4) Fermi liquid theory $\cdots\cdots$ A special case of the RG where the thermal fluctuation can be renormalized with high precision (very lucky for condensed matter physicists.)

$\longrightarrow$ Band picture

- Computers help to a certain extent in solving. However, without $\mathcal{O}(1)$ description, we can't feel that we understand the issue.

- In this lecture, we learn how we can reduce ~~the~~ many-body problem into $\mathcal{O}(1)$ problem, which is not $\Rightarrow$ guaranteed to be possible.

[1] Second Quantization and Bosons

- Second quantization is not a physical operation.
- Second quantization is just another way of describing the same things.
- Conventional description, the coordinate description, $\Psi(x_1 \cdots x_N)$ is redundant when any two particles are essentially indistinguishable.
- SQ description removes this redundancy.
- Our first target is to describe the equation and the solution of the Schrödinger eq.

$$-\frac{d}{dt}\Psi(\mathbb{X}) = H\Psi(\mathbb{X})$$

where H is an operator consisting of $\mathbb{X}$ and $\partial/\partial\mathbb{X}$. $\mathbb{X} = (x_1 \cdots x_N)$.

Conventional (coordinate) description

$$\Psi(\mathbb{X}) \qquad \mathbb{X} = (x_1\, x_2 \cdots x_N)$$

x_i .... the "coordinate" of the i-th particle.
(In general, it includes the inner degrees of freedom such as spins.)

Hilbert space of many identical particles

- Any permutation does not change the physical state.

$\Longrightarrow \quad P\Psi = \chi_P \Psi$ (In 2D, $P\Psi$ is not uniquely defined because the result of permutation may depend on the path.)

Ψ …. the wave function of N identical particles
P …. an arbitrary permutation ($P \in S_N$) ← the symmetric group
$P\Psi(\mathbb{X}) \equiv \Psi(P\mathbb{X})$
$P\mathbb{X} \equiv (x_{P(1)}, x_{P(2)}, \cdots, x_{P(N)})$
$\chi_P \in \mathbb{C} \quad |\chi_P| = 1$

- This means χ_P is a one-dim. rep. of S_N.

$\because \quad \chi_{PQ}\Psi = (PQ)\Psi = P(Q\Psi) = \chi_P \chi_Q \Psi$
$\longrightarrow \chi_{PQ} = \chi_P \chi_Q$

- There is only two 1-dim. reps. of S_N:
  the identity ($\chi_P = 1$) and the parity ($\chi_P = (-1)^P$)

- There are two kinds of particles:

Bosons $\longleftrightarrow \chi_P = 1$
Fermions $\longleftrightarrow \chi_P = (-1)^P$

※ For a more complete argument, we need continuous deformations. This leads us to consider the braid group. However, the braid group is non-trivial except in two dimensions. It means that the factor χ depends only on the permutation generated by the deformation, making the above argument valid. In 2D, there can be particles that are neighther bosons or fermions.

Product state representation

- The space of N particles can be spaned by the product states $\{\Phi_{\mathfrak{x}}(X)\}$, i.e., an arbitrary N-particle wave function can be expressed as

$$\Psi(\mathbb{X}) = \sum_{\mathfrak{x}} a_{\mathfrak{x}} \Phi_{\mathfrak{x}}(\mathbb{X}).$$

$$\Phi_{\mathfrak{x}}(\mathbb{X}) = \phi_{\varkappa_1}(\mathbb{x}_1)\phi_{\varkappa_2}(\mathbb{x}_2)\cdots\phi_{\varkappa_N}(\mathbb{x}_N)$$

$$\mathfrak{x} \equiv (\varkappa_1, \varkappa_2, \cdots, \varkappa_N)$$

- $\{\phi_\varkappa(\mathbb{x})\}$ …. "one-particle state"
an arbitrary set of orthonormal functions.

$$\langle \phi_{\varkappa'} | \phi_\varkappa \rangle \equiv \int d\mathbb{x}\, \phi^*_{\varkappa'}(\mathbb{x}) \phi_\varkappa(\mathbb{x}) = \delta_{\varkappa'\varkappa}$$

$$\sum_\varkappa \phi^*_\varkappa(\mathbb{x}') \phi_\varkappa(\mathbb{x}) = \delta(\mathbb{x}' - \mathbb{x})$$

※ Here, $\mathbb{x}$ may be taken as the position vector and $\varkappa$ as the wave number. However, when the particles have some inner degrees of freedom, such as spins, we may regard $\mathbb{x}$ as the combination of the position vector and such inner degrees of freedom. Correspondingly, $\varkappa$ may stand for $(\mathbb{k}, \sigma)$.

Space of Bosonic States

- The space of the many bosonic wave functions can be spanned by the "symmetrized" product states.

$$\Phi^{(s)}_{\mathbb{x}} \equiv C^{-1}_{\mathbb{x}} \sum_{P \in S_N} P\, \Phi_{\mathbb{x}} \qquad (\mathbb{x} = (x_1, x_2, \cdots, x_N))$$

- Here we consider the inner product in the bosonic wave functions:

$$\langle \Phi_1 | \Phi_2 \rangle = \int_{x_1 < x_2 < \cdots < x_N} d\mathbb{x}\ \Phi_1^*(\mathbb{x})\, \Phi_2(\mathbb{x}) \quad \left(\equiv \int' d\mathbb{x}\, \Phi_1^* \Phi_2\right)$$

(arbitrary order rule)

$$= \frac{1}{N!} \int d\mathbb{x}\ \Phi_1^*(\mathbb{x})\, \Phi_2(\mathbb{x})$$

- Since we can specify $\Phi^{(s)}_{\mathbb{x}}$ by the occupation number $\mathbb{n}$, instead of $\mathbb{x}$, we introduce a new symbol

$$|\mathbb{n}\rangle \equiv \Phi^{(s)}_{\mathbb{x}}$$

where $\mathbb{n} = (n_1, n_2, \cdots)$ with n_x being the number of times x appears in $\mathbb{x}$.

Exercise

Show that $C_{\varkappa} = \sqrt{N \prod_{\varkappa} (n_{\varkappa}!)}$

$$C_{\varkappa}^2 = \sum_{P,Q} \langle P \Phi_{\varkappa} | Q \Phi_{\varkappa} \rangle = N! \sum_{P} \langle \Phi_{\varkappa} | P \Phi_{\varkappa} \rangle$$

$$= \sum_{P} \int dx_1 \cdots dx_N \, \phi^*_{\varkappa_{P(1)}}(x_1) \cdots \phi^*_{\varkappa_{P(N)}}(x_N) \times \phi_{\varkappa_1}(x_1) \cdots \phi_{\varkappa_N}(x_N)$$

- The integral is 1 if $\varkappa_{P(i)} = \varkappa_i$ for all i, and 0 otherwise.
- This means that for any given $\varkappa$, and the set $I(\varkappa) \equiv \{ i \text{ s.t. } \varkappa_i = \varkappa \}$,
  $P\, I(\varkappa) = I(\varkappa)$
- For each $\varkappa$, this constraint allows $|I(\varkappa)|! = n_{\varkappa}!$ possibilities.
  Therefore we have $\prod_{\varkappa} n_{\varkappa}!$ such permutations.

$$= \prod_{\varkappa} n_{\varkappa}!$$

Creation / Annihilation Operators

- It's convenient to define the "annihilation" operator, b_x, by

$$b_x |n\rangle = \sqrt{n_x}\, |n - \mathbb{1}_x\rangle \quad (\text{definition})$$

- Fock space : In the above definition of b_x we have extended the state space from the one with the fixed N to the one with various N.
  ($H_{\text{Fock}} = H_{N=0} \oplus H_{N=1} \oplus \cdots$)
- $\mathbb{1}_x$ is the vector with only one non-zero element as the "x"-th element and the other all zeros.

- By defining $|\text{vac}\rangle = |0\rangle = \underset{\sim}{1}$
  we have, for example,

  (coordinate) (the conventional state representation)

$$b_x^{\dagger} |0\rangle = |\mathbb{1}_x\rangle = \phi_x$$

- From the definition, $b_x^{\dagger}$ must have the matrix element like

$$b_x^{\dagger} |n - \mathbb{1}_x\rangle = \sqrt{n_x}\, |n\rangle$$

or $$b_x^{\dagger} |n\rangle = \sqrt{n_x + 1}\, |n + \mathbb{1}_x\rangle$$

$$\longrightarrow [b_{x'}, b_x^{\dagger}] = \delta_{x'x}$$

- $b_{x_1}^{\dagger} b_{x_2}^{\dagger} \cdots b_{x_N}^{\dagger} |0\rangle = \left(\prod_x \sqrt{n_x!}\right) |n\rangle \left(= \sum_P P \Phi_x\right)$

- Obviously, $b_x^{\dagger} b_x |n\rangle = n_x |n\rangle$. So $\hat{n}_x = b_x^{\dagger} b_x$ is the occupation number operator.

Field Operator

- It is also convenient to define the field op.

$$\hat{\psi}(x) \equiv \sum_{\kappa} \phi_{\kappa}(x)\, b_{\kappa} \quad \left(\hat{\ } \text{ … operator in the Fock space}\right)$$

- We can show that

$$[\hat{\psi}(x), \hat{\psi}^{+}(y)] = \delta(x-y) \qquad ①$$

- We define $|x\rangle \equiv \hat{\psi}^{+}(x)|0\rangle$, then

$$|x\rangle(y) = \delta(x-y) \qquad ②$$

← remember that for each wave fn. represented in the 2nd quantization, there is always corresponding coordinate sp. representation.

- Similarly, we define

$$|\mathbb{X}\rangle \equiv \hat{\psi}^{+}(x_1)\cdots\hat{\psi}^{+}(x_N)|0\rangle.$$

Then, we have

$$|\mathbb{X}\rangle(\mathbb{Y}) = \sum_{P\in S_N} \delta(\mathbb{Y}-P\mathbb{X}) \qquad ③$$

and

$$\langle \mathbb{X}'|\mathbb{X}\rangle = \sum_{P\in S_N} \delta(\mathbb{X}'-P\mathbb{X}) \qquad ④$$

(when we have no accidental coincidence $(x_i = x_j)$)

- We use later

$$\hat{\psi}(x)|\mathbb{X}\rangle = \sum_i \delta(x-x_i)\,|\mathbb{X}^{[i]}\rangle \quad \left(\mathbb{X}^{[i]} = (x_1\cdots x_{i-1}, x_{i+1}, \cdots, x_N)\right) \qquad ⑤$$

- Proof of 1 :

$$[\hat{\psi}(x), \hat{\psi}^{+}(y)] = \sum_{\nu,\nu'} \phi_\nu(x)\phi^*_{\nu'}(y)[b_\nu b^+_{\nu'}]$$

$$= \sum_{\nu,\nu'} \phi_\nu(x)\phi^*_{\nu'}(y)\delta_{\nu\nu'}$$

$$= \sum_{\nu} \phi_\nu(x)\phi^*_{\nu}(y) = \delta(x-y)$$

- Proof of 2

$$|x\rangle(y) = \hat{\psi}^{+}(x)|0\rangle(y)$$

$$= \sum_\nu \phi^*_\nu(x)\underbrace{b^+_\nu|0\rangle}(y)$$

$$= \sum_\nu \phi^*_\nu(x)\phi_\nu(y) = \delta(x-y)$$

- Proof of 3 (for $N=2$)

$$|X\rangle(Y) = \hat{\psi}^{+}(x_1)\hat{\psi}^{+}(x_2)|0\rangle(Y)$$

$$= \sum_{\nu\nu'} \phi^*_\nu(x_1)\phi^*_{\nu'}(x_2)\, b^+_\nu b^+_{\nu'}|0\rangle(Y)$$

$$= \sum_{\nu\nu'} \phi^*_\nu(x_1)\phi^*_{\nu'}(x_2)\sum_P (P\,\Phi_{\nu\nu'})(Y)$$

$$= \sum_{\nu\nu'} \phi^*_\nu(x_1)\phi^*_{\nu'}(x_2)\left(\Phi_{(\nu\nu')}+\Phi_{(\nu'\nu)}\right)(Y)$$

$$= \sum_{\nu\nu'} \phi^*_\nu(x_1)\phi^*_{\nu'}(x_2)\left(\phi_\nu(y_1)\phi_{\nu'}(y_2) + (\nu\leftrightarrow\nu')\right)$$

$$= \delta(x_1-y_1)\delta(x_2-y_2) + \delta(x_1-y_2)\delta(x_2-y_1)$$

(Similar arguments prove $N \geq 3$.)

∘ Proof of 4

using 3, we get

$$\langle X' | X \rangle = \frac{1}{N!} \int dY \left(|X'\rangle(Y) \right)^* \left(|X\rangle(Y) \right)$$

$$= \frac{1}{N!} \int dY \sum_{PP'} \delta(Y - P'X')\, \delta(Y - PX)$$

$Y' \equiv P^{-1} Y$

$$= \frac{1}{N!} \int dY' \sum_{PP'} \delta(PY' - P'X')\, \delta(Y' - X)$$

$$= \frac{1}{N!} \sum_{PP'} \delta(PX - P'X')$$

$$= \frac{1}{N!} \sum_{PP'} \delta(X' - \underbrace{P'^{-1}P}_{Q} X)$$

$$= \frac{1}{N!} \sum_{PQ} \delta(X' - QX) = \sum_{Q} \delta(X' - QX)$$

- Proof of ⑤

$$\hat{\psi}(x)|X\rangle = \hat{\psi}(x)\hat{\psi}^+(x_1)\cdots\hat{\psi}^+(x_N)|0\rangle$$

$$= \Big([\hat{\psi}(x)\,\hat{\psi}^+(x_1)]\,\psi^+(x_2)\cdots\hat{\psi}^+(x_N) + \hat{\psi}^+(x_1)\,\hat{\psi}(x)\,\psi^+(x_2)\cdots\hat{\psi}^+(x_N)\Big)|0\rangle$$

$$= \Big([\hat{\psi}\;\hat{\psi}_1^+]\hat{\psi}_2^+\cdots\hat{\psi}_N^+ + \psi_1^+[\hat{\psi}\;\hat{\psi}_2]\hat{\psi}_3^+\cdots\hat{\psi}_N^+ + \hat{\psi}_1^+\hat{\psi}_2^+\hat{\psi}\,\hat{\psi}_3^+\cdots\hat{\psi}_N^+\Big)|0\rangle$$

$$\vdots$$

$$= \Big([\hat{\psi}\;\hat{\psi}_1^+]\psi_2^+\cdots\psi_N^+ + \cdots + \psi_1^+\psi_2^+\cdots[\hat{\psi}\;\hat{\psi}_N^+]\Big)|0\rangle$$

$$= \Big(\delta(x-x_1)\psi_2^+\cdots\psi_N^+ + \cdots + \delta(x-x_N)\psi_1^+\cdots\psi_{N-1}^+\Big)|0\rangle$$

$$= \sum_i \delta(x-x_i)|X^{[i]}\rangle$$