London Equation and Meissner Effect

Phenomenological Argument

- Lagrangean for a single particle

$$\left(\begin{array}{l} q \equiv \text{charge} \\ \mathbb{E} = \nabla\phi \\ \mathbb{B} = \nabla \times \mathbb{A} \\ \mathbb{v} \equiv \dot{\mathbb{x}} \end{array} \right)$$

$$\mathcal{L} = \frac{m}{2} \mathbb{v}^2 - q\phi + q\mathbb{v}\cdot\mathbb{A}$$

$$\mathbb{P} \equiv \frac{\partial \mathcal{L}}{\partial \mathbb{v}} = m\mathbb{v} + q\mathbb{A}$$

$$\mathcal{H}_1 = \mathbb{p}\mathbb{v} - \mathcal{L}$$

$$= \mathbb{p}\mathbb{v} - \frac{m}{2}\mathbb{v}^2 + q\phi - q\mathbb{v}\cdot\mathbb{A}$$

$$= \frac{1}{2m}(\mathbb{p} - q\mathbb{A})^2 + q\phi$$

- Electronic System and Meissner Effect

$$\mathcal{H}_1 = \sum_i \left(\frac{1}{2m}(\mathbb{p}_i + e\mathbb{A})^2 - e\phi(\mathbb{x}_i) \right)$$

$$\mathbb{j} \equiv -ne\mathbb{v} = -\frac{ne}{m}(\mathbb{p} + e\mathbb{A})$$

If the $\mathbb{p}$-term did not contribute to $\mathbb{j}$, i.e., if $\mathbb{j} \sim -\frac{ne^2}{m}\mathbb{A}$, then, the Maxwell eq.

$$\nabla \times \mathbb{B} - \frac{1}{c^2}\frac{\partial \mathbb{E}}{\partial t} = \mu_0 \mathbb{j}$$

yields $\nabla \times (\nabla \times \mathbb{A}) = -\frac{\mu_0 n e^2}{m}\mathbb{A} = -\frac{1}{\lambda^2}\mathbb{A}$ $\left(\lambda^2 \equiv \frac{m}{\mu_0 n e^2}\right)$

for its stable solution ($\frac{\partial \mathbb{E}}{\partial t} = 0$). Since

$\nabla \times (\nabla \times \mathbb{A}) = \nabla(\nabla\cdot\mathbb{A}) - \Delta\mathbb{A} = -\Delta\mathbb{A}$

in the Coulomb gauge ($\nabla\cdot\mathbb{A} = 0$), we obtain

$$\Delta\mathbb{A} = \lambda^{-2}\mathbb{A}.$$

$\Rightarrow$ $\mathbb{A}$ damps exponentially with the penetration length λ.

In superconductors, as we see below, something similar is happening; the magnetic field is absent deep inside the superconductor. This is called Meissner effect. In other words, the external field is cancelled by some "a counter" field. This counter field is generated by a persistent circular current near the surface. In order for such a state to be a thermodynamically stable equillibrium, the electronic registivity must be zero because otherwise the persistent current must die out due to the dissipation.

Thus, by establishing the London equation, we can show the two characterising properties of superconductivity, the Meissner effect and the zero registivity!

That's what we'll do in the following.

Quantum Field Theoretical Derivation of London Equation

- We want to prove that, in superconducting state,

$$\langle \hat{\mathring{j}} \rangle = -\frac{n_s e^2}{m} \mathbb{A} \quad (n_s > 0) \quad \text{(London equation)}$$

which leads to the Meissner effect. As long as $n_s > 0$, it doesn't have to be exactly equal to n.

- The current density operator $\mathring{j}$.

Classically, $\mathring{j} = -en v = -e\frac{1}{m} n(\mathbb{p} + e\mathbb{A})$

which suggests $\hat{\mathring{j}} = -\frac{e}{m} \hat{\psi}^+ \left(\frac{\hbar}{i} \nabla + e\mathbb{A} \right) \hat{\psi}$.

However, the first term $-\frac{e}{m} \hat{\psi}^+ \frac{\hbar}{i} (\nabla \hat{\psi})$ is not Hermitian, To make it Hermitian we modify it to

$$-\frac{e}{m} \frac{1}{2} \left(\hat{\psi}^+ \left(\frac{\hbar}{i} \nabla \hat{\psi} \right) + \text{h.c.} \right)$$

$$= \frac{e\hbar}{2m} \frac{1}{i} \left((\nabla \hat{\psi}^+) \hat{\psi} - \hat{\psi}^+ (\nabla \psi) \right)$$

Thus we obtain

$$\hat{\mathring{j}} = \hat{\mathring{j}}_p + \hat{\mathring{j}}_d$$

$$\hat{\mathring{j}}_p \equiv \frac{e\hbar}{2mi} \left((\nabla \hat{\psi}^+) \hat{\psi} - \hat{\psi}^+ (\nabla \hat{\psi}) \right)$$

$$\hat{\mathring{j}}_d \equiv -\frac{e^2}{m} \mathbb{A} \hat{\psi}^+ \hat{\psi}$$

- So our task is to compute $\mathring{j}_p \equiv \langle \hat{\mathring{j}}_p \rangle$ in

$$\mathring{j} = \mathring{j}_p + \mathring{j}_d = \mathring{j}_p - \frac{ne^2}{m}A \quad \text{——} \; (*)$$

$$\left(\mathring{j}_d \equiv \langle \hat{\mathring{j}}_d \rangle = -\frac{e^2}{m}\langle \hat{\psi}^+ \hat{\psi} \rangle A = -\frac{ne^2}{m}A \right)$$

In the following, we will show that the linear responce of $\mathring{j}_p$ to A is zero, i.e., $\mathring{j}_p = o(A)$ in the $q \to 0$ limit. Therefore, even if A is finite, as long as it's small enough, the second term in $(*)$ is dominating, leading to Meissner effect.

$$\begin{aligned}
\hat{V} &\equiv \hat{\mathcal{H}}(A) - \hat{\mathcal{H}}(0) \\
&= \int dx \frac{1}{2m} \hat{\psi}^+(x)\left((p + eA)^2 - p^2 \right)\psi(x) \\
&= \int dx \frac{e}{2m} \hat{\psi}^+(x)\left(p \cdot A + A \cdot p \right)\hat{\psi}(x) \\
&= \frac{e\hbar}{2mi} \int dx \, \hat{\psi}^+(x)\left(\nabla \cdot A + A \cdot \nabla \right)\hat{\psi}(x) \\
&= \frac{e\hbar}{2mi} \int dx \left(-\nabla\hat{\psi}^+(x)) \cdot A \hat{\psi}(x) + \hat{\psi}^+(x) A \cdot (\nabla\hat{\psi}(x) \right) \\
&= -\int dx \, A(x) \cdot \hat{\mathring{j}}_p(x)
\end{aligned}$$

Because of the linear response theorem, it follows

$$\mathring{j}_p(x) = \int dx' \int_0^\beta d\tau \, \langle \underbrace{\hat{\mathring{j}}_p(x,\tau) \, \hat{\mathring{j}}_p^+(x',0)}_{3\times 3 \text{ matrix}} \rangle_0 \, A(x') \quad \text{—} \; (1)$$

$$\hat{\mathring{j}}_p(x,\tau) \equiv e^{\tau\mathcal{H}_0} \, \hat{\mathring{j}}_p(x) \, e^{-\tau\mathcal{H}_0}$$

∘ Fourier transformation

$$\hat{\mathring{J}}_p(x) \equiv i\frac{e\hbar}{2m}\sum_\sigma\left(\hat{\psi}^+_\sigma(\nabla\hat{\psi}_\sigma) - (\nabla\hat{\psi}^+_\sigma)\hat{\psi}_\sigma\right)$$

$$\hat{\psi}_\sigma(x) = \frac{1}{\sqrt{L^d}}\sum_k e^{ikx} c_{k\sigma}$$

$$\hat{\mathring{J}}_p(x) = \frac{1}{\sqrt{\Lambda}}\sum_q e^{iqx}\hat{\mathring{J}}_p(q) \qquad \left(\Lambda \equiv \frac{L^d}{a^d}\right)$$

$$\hat{\mathring{J}}_p(q) \equiv -\frac{e\hbar}{2ma^d}\frac{1}{\sqrt{\Lambda}}\sum_{k\sigma}(2k+q)c^+_{k\sigma}c_{k+q\sigma}$$

$$\hat{V} = -a^d\sum_k \mathring{J}^+_p(k)A(k)$$

The linear responce :

$$\langle\hat{\mathring{J}}^\alpha_p(q)\rangle = \frac{1}{\Lambda}\int dq' A^\beta(q')\chi_{j^\beta_p(q)j^{\alpha+}_p(q')}$$

$$\chi_{XY} \equiv \int_0^\beta d\tau\langle X(\tau)Y(0)\rangle_0 \quad \begin{pmatrix} \Delta\mathcal{H} = -\eta X \\ \langle Y\rangle = \chi_{XY}\eta\end{pmatrix}$$

$$\rightarrow \quad \mathring{J}_p(k) = \sum_{k'} S(k,k')A(k')$$

$$S(k,k') \equiv a^d\int_0^\beta d\tau\langle\hat{\mathring{J}}_p(k,\tau)\hat{\mathring{J}}^+_p(k',0)\rangle$$

- Current-current correlation function

Here we remember that the BCS state is the vacuum of Bogoliubov fermions that are related to the original fermions by

$$\begin{pmatrix} c_{k\uparrow} \\ c^{+}_{\bar{k}\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & v_k \\ -v_k & u_k \end{pmatrix} \begin{pmatrix} \alpha_k \\ \beta^{+}_k \end{pmatrix} \qquad \begin{pmatrix} u_k = \cos\theta_k \\ v_k = \sin\theta_k \\ \tan 2\theta_k = \frac{\Delta}{\xi_k} \end{pmatrix}$$

We can rewrite the $\hat{j}^{\,o}_p$ in terms of α_k and β_k as follows.

$$\hat{j}^{\,o}_p(q) = -\frac{e\hbar}{2ma^d}\frac{1}{\sqrt{\Omega}}\sum_{k\sigma}{}'(2k+q)\,c^{+}_{k\sigma}c_{k+q\,\sigma}$$

$$= -\frac{e\hbar}{2ma^d}\frac{1}{\sqrt{\Omega}}\sum_{k}{}'\left((2k+q)\left(c^{+}_{k\uparrow}c_{k+q\uparrow}+c^{+}_{k\downarrow}c_{k+q\downarrow}\right)\right)$$

$k+q \to \bar{k}$ in the 2nd term

$$= -\frac{e\hbar}{2ma^d}\frac{1}{\sqrt{\Omega}}\sum_{k}{}'\left((2k+q)\left(c^{+}_{k\uparrow}c_{k+q\uparrow}-c^{+}_{\bar{k}+\bar{q}\downarrow}c_{\bar{k}\downarrow}\right)\right)$$

$(\bar{k} \equiv -k)$

$$S(q'q) = a^d\int_0^\beta d\tau\,\langle \hat{j}^{\,o}_p(q',\tau)\,\hat{j}^{\,o\,+}_p(q,0)\rangle_0$$

$$\langle \hat{j}^{\,o}_p(q'\tau)\,\hat{j}^{\,o\,+}_p(q,0)\rangle_0 = -\left(\frac{e\hbar}{2ma^d}\right)^2\frac{1}{\Omega}\sum_{k'k}{}'(2k'+q')(2k+q)^T$$

$$\times\left\langle\left(c^{+}_{k'\uparrow}(\tau)c_{k'+q'\uparrow}(\tau)-c^{+}_{\bar{k}'+\bar{q}'\downarrow}(\tau)c_{\bar{k}'\downarrow}(\tau)\right)\left(c^{+}_{k+q\uparrow}c_{k\uparrow}-c^{+}_{\bar{k}\downarrow}c_{\bar{k}+\bar{q}\downarrow}\right)\right\rangle \quad —①$$

Here we encounter seemingly complicated expression!

$$\langle\cdots\rangle_0 = \langle c^{+\,(\tau)}_{k'\uparrow}c^{(\tau)}_{k'+q'\uparrow}c^{+}_{k+q\uparrow}c_{k\uparrow} + \text{3 other terms}\rangle_0 \quad —②$$

To proceed further, we'll use Wick's theorem.
For example, for the first quartet in ②,

$$\langle c^{\dagger}_{k'\uparrow}(\tau)\, c_{k'+q'\uparrow}(\tau)\, c^{\dagger}_{k+q\uparrow}\, c_{k\uparrow} \rangle_0$$

$$= \langle c^{\dagger}_{k'\uparrow}(\tau)\, c_{k'+q'\uparrow}(\tau)\rangle_0 \langle c^{\dagger}_{k+q\uparrow}\, c_{k\uparrow}\rangle_0$$
$$- \langle c^{\dagger}_{k'\uparrow}(\tau)\, c^{\dagger}_{k+q\uparrow}\rangle_0 \langle c_{k'+q'\uparrow}(\tau)\, c_{k\uparrow}\rangle_0$$
$$+ \langle c^{\dagger}_{k'\uparrow}(\tau)\, c_{k\uparrow}\rangle_0 \langle c_{k'+q'\uparrow}(\tau)\, c^{\dagger}_{k+q\uparrow}\rangle_0 \qquad \text{—— ③}$$

Here, we can neglect the 1st term because if we collect the 1st terms in the 4 quartets in ②, and sum them over k and k' and Fourier trans. back $q \to x$, the result is $\langle \hat{j}^{0}_{p}(x,\tau)\rangle_0 \langle \hat{j}^{0\dagger}_{p}(x,0)\rangle_0 = 0$ (since $\langle \hat{j}_p(x,\tau)\rangle_0 = 0$ due to the invariance with respect to space inversion).
So, in what follows we'll only consider "cross" terms.

In addition, since $\langle c_{k\sigma}\, c_{k'\sigma'}\rangle \neq 0$ only when $k' = -k$ and $\sigma' = -\sigma$, and $\langle c^{\dagger}_{k\sigma}\, c_{k'\sigma'}\rangle \neq 0$ only when $k' = k$ and $\sigma' = \sigma$ due to the Bogoliubov mixing, the 2nd in ③ is zero and the 3rd term is non-zero only when $k = k'$ and $q = q'$. The same applies to the other quartets in ②. Thus we have

$$S(q, q') = \delta_{q,q'}\, S(q)$$

$$S(q) = a^d \int_0^{\beta} \langle \hat{j}^{0}_{p}(q,\tau)\, \hat{j}^{0\dagger}_{p}(q,0)\rangle_0$$

where in computing $\langle \hat{j}^{0}_{p}\, \hat{j}^{0\dagger}_{p}\rangle_0$, we have only to consider $k = k'$ terms.

Therefore,

$$\langle \hat{j}_p(q,\tau)\,\hat{j}_p^{\dagger}(q,0)\rangle_0$$

$$= \left(\frac{e\hbar}{2mad}\right)^2 \frac{1}{\Lambda} \sum_k{}' (2k+q)(2k+q)^{\dagger} \langle Q_{kq}(\tau)\, Q^{\dagger}_{kq}(0)\rangle_0$$

$$Q_{kq} \equiv c^{\dagger}_{k\uparrow} c_{k+q\uparrow} - c^{\dagger}_{\bar{k}+\bar{q}\downarrow} c_{\bar{k}\downarrow}$$

$$= (u_k \alpha_k^{\dagger} + v_k \beta_k)(u_{k+q}\alpha_{k+q} + v_{k+q}\beta^{\dagger}_{k+q}) - (-v_{k+q}\alpha_{k+q} + u_{k+q}\beta^{\dagger}_{k+q})(-v_k\alpha^{\dagger}_k + u_k\beta_k)$$

$$= \underbrace{(u_k u_{k+q} + v_{k+q} v_k)}_{\to 1}(\alpha^{\dagger}_k \alpha_{k+q} - \beta^{\dagger}_{k+q}\beta_k)$$

$$+ \underbrace{(u_k v_{k+q} - u_{k+q} v_k)}_{\xrightarrow{q \to 0} 0}(\alpha^{\dagger}_k \beta^{\dagger}_{k+q} - \beta_k \alpha_{k+q})$$

($u_k = u_{\bar{k}}$, $v_k = v_{\bar{k}}$ has been used.)

Eventually, we'll be most interested in the long-range behavior which is governed by small-q part of $\mathcal{S}(q)$.
In the limit $q \to 0$, we may replace Q_{kq} by

$$Q_{kq} \underset{q\to 0}{\approx} \alpha^{\dagger}_k \alpha_{k+q} - \beta^{\dagger}_{k+q}\beta_k$$

Then, we have

$$\langle Q_{kq}(\tau) Q^{\dagger}_{kq}(0)\rangle_0 \sim \langle \alpha^{\dagger}_k(\tau)\alpha_{k+q}(\tau)\,\alpha^{\dagger}_{k+q}(0)\alpha_k(0)\rangle_0$$

$$+ \langle \beta^{\dagger}_{k+q}(\tau)\beta_k(\tau)\,\beta^{\dagger}_k(0)\,\beta_{k+q}(0)\rangle_0$$

Generally, $\int_0^\beta d\tau \langle X(\tau) Y(0) \rangle_0 = \frac{1}{Z_0} \sum_{\mu\nu} \chi_{\mu\nu} X_{\mu\nu} Y_{\mu\nu}$

$$\chi_{\mu\nu} = -\frac{e^{-\beta E_\mu} - e^{-\beta E_\nu}}{E_\mu - E_\nu} = e^{-\beta E_\mu} \frac{1 - e^{-\beta \Delta_{\nu\mu}}}{\Delta_{\nu\mu}}$$

($\Delta_{\mu\nu} \equiv E_\nu - E_\mu$; E_μ = the eigenvalue of the non-perturbative Hamiltonian (i.e. $\mathcal{H}_{BCS}$ with $A=0$))

$$\int_0^\beta d\tau \langle (\alpha_k^+ \alpha_{k+q})(\tau) (\alpha_{k+q}^+ \alpha_k) \rangle_0 = \int_0^\beta d\tau \langle (\beta_{k+q}^+ \beta_{k})(\tau) (\beta_k^+ \beta_{k+q}) \rangle_0$$

$$= \frac{1}{Z_0} \sum_\mu e^{-\beta E_\mu} \frac{1 - e^{-\beta(E_{k+q} - E_k)}}{E_{k+q} - E_k} \langle \mu | \alpha_k^+ \alpha_{k+q} \alpha_{k+q}^+ \alpha_k | \mu \rangle_0$$

$$= \frac{1}{Z_0} \frac{1 - e^{-\beta(E_{k+q} - E_k)}}{E_{k+q} - E_k} \sum_\mu e^{-\beta E_\mu} \langle \mu | \quad \text{''} \quad | \mu \rangle_0$$

$$= \frac{1 - e^{-\beta(E_{k+q} - E_k)}}{E_{k+q} - E_k} f_k (1 - f_{k+q})$$

$$= \frac{f_k(1 - f_{k+q}) - (1 - f_k) f_{k+q}}{E_{k+q} - E_k} \quad \left(\begin{array}{l} e^{+\beta E_k} f_k = 1 - f_k \\ e^{-\beta E_{k+q}} (1 - f_{k+q}) = f_{k+q} \end{array} \right)$$

$$= -\frac{f_{k+q} - f_k}{E_{k+q} - E_k} \approx -\frac{\partial f}{\partial E} \overset{\beta \to 0}{\approx} \delta(E) \left(\begin{array}{c} \text{Dirac} \\ \text{delta} \end{array} \right)$$

$$\therefore \; S(q' q) = \delta_{q'q} \, S(q)$$

$$S(q) \overset{q \to 0}{\approx} a^d \left(\frac{e\hbar}{2ma^d} \right)^2 \frac{1}{\Omega} \sum_k{}' (2k + q)(2k + q)^T \, 2 \times \left(-\frac{\partial f}{\partial E} \right)$$

(Since there is no state at $E=0$ due to $\Delta > 0$, $S(q) \sim 0$ is already obvious.)

$$S(q) = \left(\frac{e\hbar}{2m} \right)^2 \int' \frac{dk}{(2\pi)^d} \, 8 \, k k^T \left(-\frac{\partial f}{\partial E} \right)$$

$$= \left(\frac{e\hbar}{2m} \right)^2 \int' \frac{dk}{(2\pi)^d} \, 8 k^2 \left(\frac{1}{4\pi} \int_{|n|=1} d\Omega^{d-1} \, n n^T \right) \left(-\frac{\partial f}{\partial E} \right)$$

$$S(q) = \left(\frac{e\hbar}{2m}\right)^2 \int' \frac{d^3k}{(2\pi)^3}\, 8k^2\, \frac{1}{4\pi}\left(\frac{4\pi}{3} I\right)\left(-\frac{\partial f}{\partial E}\right)$$

$$= \underbrace{\left(\left(\frac{e\hbar}{m}\right)^2 \cdot \frac{2}{3} \int' \frac{d^3k}{(2\pi)^3}\, k^2 \left(-\frac{\partial f}{\partial E}\right)\right)}_{\sigma(q)} I$$

① Normal state $\left(\Delta = 0 \quad E = \epsilon = \frac{\hbar^2 k^2}{2m}\right)$

$$\frac{\partial f}{\partial E} = \frac{\partial f}{\partial \epsilon} = \left(\frac{d\epsilon}{dk}\right)^{-1} \frac{\partial f}{\partial k} = \left(\frac{\hbar^2}{m} k_F\right)^{-1} \delta(k - k_F)$$

$$\longrightarrow \sigma(q) = \left(\frac{e\hbar}{m}\right)^2 \frac{2}{3} \int \frac{d^3k}{(2\pi)^3} k^2 \left(\frac{m}{\hbar^2 k_F}\right) \delta(k - k_F)$$

$$= \frac{ne^2}{m} \quad \left(\because 2 \times \frac{4\pi}{3} k_F^3 \Big/ \left(\frac{2\pi}{L}\right)^3 = N\right)$$

$$\longrightarrow j = \frac{ne^2}{m} A - \frac{ne^2}{m} A = 0 \quad \left(\text{no persistent current}\right)$$

② Superconducting state $(\Delta > 0)$

$$\sigma(q) \underset{q \to 0}{\sim} \left(\frac{e\hbar}{m}\right)^2 \frac{2}{3} \int' \frac{d^3k}{(2\pi)^3} k^2 \left(-\frac{\partial f}{\partial E}\right)$$

$$\xrightarrow[T \to 0]{} 0$$

This is because $-\frac{\partial f}{\partial E} = -\frac{\partial}{\partial E} \frac{1}{e^{\beta E} + 1}$

is like the Dirac δ-function at $E = 0$,
whereas no state has $E = 0$ because $|E| \geq \Delta > 0$.

Thus we've proved $j \approx 0 - \frac{ne^2}{m} A \quad (q \to 0 \;\; T \to 0)$
which means London's equation.