Lecture 9: Perturbative Renormalization Group

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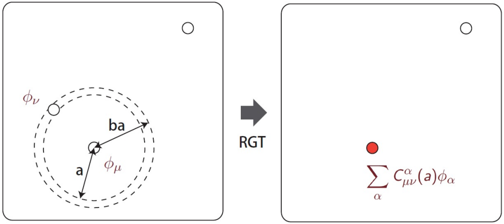
In this lecture, we see ...

- When there is a fixed point and we know its OPE, by a perturbative argument, we can derive a set of equations describing RG flow around it. (Then, we can study the behavior of other fixed points in its vicinity, as we will discuss in the next lecture.)
- We can obtain the renormalized Hamiltonian up to the 2nd order (or more if we try harder) in the case of GFP, which is the lowest non-trivial order.

[9-1] General perturbative RG

- We decompose the field operator into the high-frequency component and the low-frequency component.
- Tracing out the high-fruquency component, followed by rescaling, yields the RG flow equations.
- In the RGT from the scale a to ab ($b = 1 + \delta$), the product of two scaling operators within the distance of a, gives rise to new perturbative terms through OPE, which contributes non-linear terms in the RG flow equation.



Expanding the Hamiltonian around a fixed point

- Consider some fixed-point Hamiltonian, $\mathcal{H}_a^*$, with short-distant cut-off (lattice constant) a, and consider a general Hamiltonian expressed in terms of the scaling-operators at $\mathcal{H}_a^*$:

$$\mathcal{H}_a \equiv \mathcal{H}_a^* + \Delta\mathcal{H}_a \qquad \left(\Delta\mathcal{H}_a \equiv \sum_\alpha g_\alpha \int_a d\mathbf{x}\, \phi_\alpha(\mathbf{x}) \right)$$

where ϕ_α is the scaling operator at $\mathcal{H}^*$ with the dimension x_α.

$$\phi_\alpha(\mathbf{x}) \;\rightarrow\; \acute{\phi}_\alpha(\acute{\mathbf{x}}) = \mathcal{R}_b \phi_\alpha(\mathbf{x}) = b^{x_\alpha} \phi_\alpha(\mathbf{x})$$

RGT to the expansion

- Let us carry out the general program of RG: (i) partial trace, and (ii) rescaling.
- We introduce the ultra-violet cut-off in the form of the restriction on the integral region in (i) that no two operators cannot be within the mutual distance a.
- By the partial trace, we will shift the cut-off length a to $\acute{a} \equiv e^{\lambda} a \approx (1 + \lambda) a$.
- Then, the partial trace is equivalent to application of the OPE to every pair of operators that come within the mutual distance of $\acute{a}$, and taking the summation with respect to the relative position of the two (This yields the factor $V_d(\acute{a}^d - a^d) \approx V_d d \lambda a^d$, where V_d is the volume of unit sphere.).

The partial trace (0th order term)

- By denoting the partial trace by Tr', the perturbative expansion becomes

$$\mathrm{Tr}' e^{-\mathcal{H}_a^* - \Delta\mathcal{H}_a} = \mathrm{Tr}' \left\{ e^{-\mathcal{H}_a^*} \left(1 - \Delta\mathcal{H}_a + \frac{1}{2}(\Delta\mathcal{H}_a)^2 - \cdots \right) \right\}$$
$$\left(\equiv e^{-\tilde{\mathcal{H}}_{ab}(\phi^l)} \right)$$

- We define Z_h and $\tilde{\mathcal{H}}_{ab}^*$ by

$$\text{(0th order term)} = \mathrm{Tr}' e^{-\mathcal{H}_a^*(\phi)} = Z_h \times e^{-\tilde{\mathcal{H}}_{ab}^*(\phi^l)}. \tag{1}$$

where the superscript l in ϕ^l is symbolic and reminder of the restriction that two operators cannot come closer than $\acute{a}$. (We also demand that $\tilde{\mathcal{H}}_{ab}^*$ will become back to $\mathcal{H}_a^*$ after the rescaling, because $\mathcal{H}_a^*$ is the fixed-point Hamiltonian.)

The partial trace (1st order term)

$$e^{-\tilde{\mathcal{H}}_a} = \mathrm{Tr}'\left\{e^{-\mathcal{H}_a^*}\left(1 - \Delta\mathcal{H}_a + \frac{1}{2}(\Delta\mathcal{H}_a)^2 - \cdots\right)\right\} \qquad \left(\Delta\mathcal{H}_a \equiv \sum_\alpha g_\alpha \int_a d\mathbf{x}\,\phi_\alpha(\mathbf{x})\right)$$

- Because of the absense of interaction, the 1st order term is easy:

$$\begin{aligned}
(\text{1st-order term}) &= -\,\mathrm{Tr}'e^{-\mathcal{H}_a^*(\phi)}\sum_\alpha g_\alpha \int_a d\mathbf{x}\,\phi_\alpha(\mathbf{x}) \\
&\overset{(*)}{=} -Z_h e^{-\tilde{\mathcal{H}}_{ab}^*(\phi')}\sum_\alpha g_\alpha \int_{ab} d\mathbf{x}\,\phi'_\alpha(\mathbf{x}) \\
&= -Z_h e^{-\tilde{\mathcal{H}}_{ab}^*(\phi')}\Delta\mathcal{H}_{ab}(\phi')
\end{aligned} \qquad (2)$$

In $(*)$, we have used

$$\mathrm{Tr}'\left[e^{-\mathcal{H}_a^*(\phi)}Q(\phi)\right] = Z_h e^{-\tilde{\mathcal{H}}_{ab}^*(\phi')}Q(\phi')$$

The partial trace (2nd order term)

$$e^{-\tilde{\mathcal{H}}_a} = \mathop{\mathrm{Tr}}_{\phi^h} \left\{ e^{-\mathcal{H}_a^*} \left(1 - \Delta\mathcal{H}_a + \frac{1}{2}(\Delta\mathcal{H}_a)^2 - \cdots \right) \right\} \qquad \left(\Delta\mathcal{H}_a \equiv \sum_\alpha g_\alpha \int_a d\mathbf{x}\, \phi_\alpha(\mathbf{x}) \right)$$

- For the 2nd-order term, we use OPE:

(2nd-order term)

$$= \frac{1}{2} \sum_{\alpha\beta} g_\alpha g_\beta \,\mathrm{Tr}' \left(e^{-\tilde{\mathcal{H}}_a^*} \int_a d\mathbf{x}d\mathbf{y}\, \phi_\alpha(\mathbf{x})\phi_\beta(\mathbf{y}) \right)$$

$$\mathrm{Tr}' \left(e^{-\tilde{\mathcal{H}}_a^*} \int_a d\mathbf{x}d\mathbf{y}\, \phi_\alpha(\mathbf{x})\phi_\beta(\mathbf{y}) \right) \Big/ \mathrm{Tr}' \left(e^{-\tilde{\mathcal{H}}_a^*} \right)$$

$$= \int_{|\mathbf{x}-\mathbf{y}|>ab} d\mathbf{x}d\mathbf{y}\, \phi_\alpha^l(\mathbf{x})\phi_\beta^l(\mathbf{y}) + \int_{a<|\mathbf{x}-\mathbf{y}|<ab} d\mathbf{x}d\mathbf{y}\, \phi_\alpha(\mathbf{x})\phi_\beta(\mathbf{y})$$

$$\approx \underbrace{\int_{ab} d\mathbf{x}d\mathbf{y}\, \phi_\alpha^l(\mathbf{x})\phi_\beta^l(\mathbf{y})}_{\text{"trivial term"}} + \underbrace{\int_{a<|\mathbf{x}-\mathbf{y}|<ab} d\mathbf{x}d\mathbf{y}\, \phi_\alpha(\mathbf{x})\phi_\beta(\mathbf{y})}_{\text{"collision term"}}$$

OPE for the collesion term

- For the collision term, we use OPE:

$$
\begin{aligned}
&\int_{a<|\mathbf{x}-\mathbf{y}|<ab} d\mathbf{x}d\mathbf{y}\,\phi_\alpha(\mathbf{x})\phi_\beta(\mathbf{y}) \\
&\quad\approx \int_{a<|\mathbf{x}-\mathbf{y}|<ab} d\mathbf{x}d\mathbf{y}\,\sum_\mu \frac{c_{\alpha\beta}^{\mu}}{a^{x_\alpha+x_\beta-x_\gamma}}\phi_\mu^l(\mathbf{x}) \\
&\quad= \int_{ab} d\mathbf{x}\,V_d((ab)^d - a^d)\sum_\mu \frac{c_{\alpha\beta}^{\mu}}{a^{x_\alpha+x_\beta-x_\gamma}}\phi_\mu^l(\mathbf{x}) \\
&\quad= V_d(b^d-1)\sum_\mu c_{\alpha\beta}^{\mu} a^{y_\alpha+y_\beta-y_\mu}\int_{ab} d\mathbf{x}\,\phi_\mu^l(\mathbf{x})
\end{aligned}
$$

The 2nd order term

- Putting together, the 2nd order term becomes

$$
\begin{aligned}
&(\text{2nd-order term}) \\
&= Z_h e^{-\tilde{\mathcal{H}}^*_{ab}} \frac{1}{2} \sum_{\alpha\beta} g_\alpha g_\beta \left\{ \int_{ab} d\mathbf{x} d\mathbf{y}\, \phi^l_\alpha(\mathbf{x}) \phi^l_\beta(\mathbf{y}) \right. \\
&\qquad \left. + V_d (b^d - 1) \sum_\mu c^\mu_{\alpha\beta} a^{y_\alpha + y_\beta - y_\mu} \int_{ab} d\mathbf{x}\, \phi^l_\mu(\mathbf{x}) \right\} \\
&= Z_h e^{-\tilde{\mathcal{H}}^*_{ab}} \left(\frac{1}{2} (\Delta \mathcal{H}_{ab}(\phi^l))^2 - \Delta \mathcal{H}^{(\mathrm{int})}_{ab} \right)
\end{aligned}
$$

where

$$
\Delta \mathcal{H}^{(\mathrm{int})}_{ab} \equiv -\frac{1}{2} \sum_{\alpha\beta\mu} g_\alpha g_\beta V_d (b^d - 1) \sum_\mu c^\mu_{\alpha\beta} a^{y_\alpha + y_\beta - y_\mu} \int_{ab} d\mathbf{x}\, \phi^l_\mu(\mathbf{x})
$$

Summary of partial trace

- Finally, the partial trace results in

$$\mathrm{Tr}\,' e^{-\mathcal{H}_a^*(\phi)-\Delta\mathcal{H}_a(\phi)} \approx Z_h e^{-\tilde{\mathcal{H}}_{ab}^*(\phi^l)-\Delta\mathcal{H}_{ab}(\phi^l)-\Delta\mathcal{H}_{ab}^{(\mathrm{int})}(\phi^l)}$$

- Therefore, our Hamiltonian after the partial trace is

$$\begin{aligned}\tilde{\mathcal{H}}_{ab}(\phi^l) &= \tilde{\mathcal{H}}_{ab}^*(\phi^l) + \Delta\mathcal{H}_{ab}(\phi^l) + \Delta\mathcal{H}_{ab}^{(\mathrm{int})}(\phi^l) \\ &= \tilde{\mathcal{H}}_{ab}^*(\phi^l) + \sum_\mu g_\mu \int_{ab} d\mathbf{x}\,\phi_\mu^l(\mathbf{x}) \\ &\quad - \frac{1}{2}\sum_{\mu\alpha\beta} g_\alpha g_\beta V_d (b^d - 1) c_{\alpha\beta}^\mu a^{y_\alpha+y_\beta-y_\mu} \int_{ab} d\mathbf{x}\,\phi_\mu^l(\mathbf{x}) \\ &= \tilde{\mathcal{H}}_{ab}^*(\phi^l) + \sum_\mu \tilde{g}_\mu \int_{ab} d\mathbf{x}\,\phi_\mu^l(\mathbf{x})\end{aligned}$$

where $\tilde{g}_\mu \equiv g_\mu - \frac{1}{2}\sum_{\mu\alpha\beta} g_\alpha g_\beta V_d (b^d - 1) c_{\alpha\beta}^\mu a^{y_\alpha+y_\beta-y_\mu}$

Rescaling

- By $\acute{\mathbf{x}} \equiv b^{-1}\mathbf{x}$ and $\acute{\phi}_\mu(\acute{\mathbf{x}}) \equiv b^{x_\mu}\phi^l_\mu(\mathbf{x})$,

$$\acute{\mathcal{H}}_a(\acute{\phi}) = \mathcal{H}^*_a(\acute{\phi}) + \sum_\mu \tilde{g}_\mu \int_a d\acute{\mathbf{x}}\, b^{y_\mu}\acute{\phi}_\mu(\acute{\mathbf{x}})$$

$$\Rightarrow \quad \acute{g}_\mu = b^{y_\mu}\tilde{g}_\mu = b^{y_\mu}\left(g_\mu - \frac{1}{2}\sum_{\alpha\beta} c^\mu_{\alpha\beta} g_\alpha g_\beta V_d (b^d - 1) a^{y_\alpha + y_\beta - y_\mu} \right).$$

- By absorbing the factor $\frac{d}{2}V_d a^{y_\mu}$ in the definition of g_μ and $\acute{g}_\mu$,,

$$\acute{g}_\mu = b^{y_\mu} \times \left(g_\mu - \sum_{\alpha\beta} c^\mu_{\alpha\beta} g_\alpha g_\beta \frac{(b^d - 1)}{d} \right)$$

- By rewriting this equation using $\lambda \equiv \log b$, we finally obtain

$$\frac{dg_\mu}{d\lambda} = y_\mu g_\mu - \sum_{\alpha\beta} c^\mu_{\alpha\beta} g_\alpha g_\beta + O(g^3)$$

[9-2] Perturbative RG around GFP

- The criticality of the Ising model in $d > 4$ is controled by the Gaussian fixed-point, though the critical behavior is modified by the dangerously irrelevant field.
- For $d < 4$, the Gaussian fixed-point is not stable w.r.t. the scaling operator ϕ_4. This motivates us to look for another fixed point by examining the perturbative RG flow around the Gaussian fixed point.

Critical property of the Ising model above 4-dimensions

- Consider the ϕ^4 model.

$$\mathcal{H} = \int d\mathbf{x} \left(|\nabla\phi|^2 + t\phi^2 + u\phi^4 - h\phi \right)$$

- Let us consider the ϕ^2 and ϕ^4 terms as the perturbation to the Gaussian fixed point (GFP). Then, it is natural to express the Hamiltonian in terms of scaling operators at the GFP.

$$\mathcal{H} = \int d\mathbf{x} \left(|\nabla\phi|^2 + t\phi_2 + u\phi_4 - h\phi \right)$$

- The scaling eigenvalues for these terms are

$$x_2 = 2x = d - 2 \;\Rightarrow\; y_2 = d - x_2 = 2$$
$$x_4 = 4x = 2(d - 2) \;\Rightarrow\; y_4 = d - x_4 = 4 - d.$$

- Since ϕ_4 is irrelevant if $d > 4$, the critical behavior of the ϕ^4 model (and therefore the Ising model as well) is described by the GFP.

Dangerous irrelevant operator for $d > 4$

- According to the general argument (see Lecture 7), the spontaneous magnetization should have the singularity like

$$m \propto L^{-d+y_h} = L^{-x_h} \propto (t^{-\frac{1}{y_t}})^{-x_h} = t^{\frac{d-2}{4}}. \quad \text{(wrong)}$$

- However, we saw that the mean-field theory correctly describes the critical behavior for $d > 4$ (Ginzburg criterion), which means that

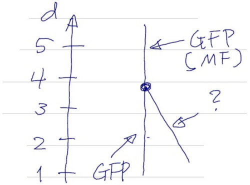
$$m \propto t^{\frac{1}{2}}. \quad \text{(correct)}$$

- This apparent contradiction comes from the nature of the irrelevant field u. Specifically, since the ϕ^4 model at or below the critical point ($t \leq 0$) is not well-defined when $u = 0$, we cannot simply put $u = 0$ in the scaling form as we did in the general argument.

Perturbative RG around GFP

- We have derived the general RG flow equation around a fixed-point.

$$\frac{dg_\mu}{d\lambda} = y_\mu g_\mu - \sum_{\alpha\beta} c^\mu_{\alpha\beta} g_\alpha g_\beta \qquad (3)$$



- If we apply this to GFP, we immediately notice that, for $d > 4$, there is only one relevant field t, implying that the GFP is the controling fixed point.
- Even below four dimensions, we may be able to obtain a new fixed point from (3) if it is near the GFP.
- In other words, we may try to find g_μ that makes the r.h.s. of (3) zero and deduce its properties from (3). (Next lexture)

Summary

- We have derived a set of equations describing RG flow around a given fixed point.
- We can obtain the renormalized Hamiltonian up to the 2nd order (or more if we try harder) in the case of GFP, which is the lowest non-trivial order.
- Above four dimensions, the critical point is controled by the Gaussian fixed point.
- However, the dangerously irrelevant field, u, modifies the critical behaviors to mean-field like.
- Below four dimensions, the critical point is not controled by the Gaussian fixed point because u becomes relevant.
- We may be able to find the "true" fixed point by analyzing the RG flow equation. (Next lecture)

Exercise

- We saw an apparent contradiction between the general scaling argument and the mean-field behaviors expected from the Ginzburg criterion. Think of a scaling form of the singular part of the free energy that obeys the scaling properties expected from the general argument, and, at the same time, produces the correct mean-field critical behaviors.