Lecture 6: General Framework of Renormalization Group — Fixed Points and Scaling Operators

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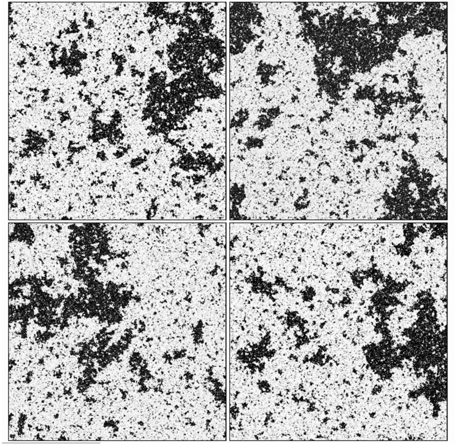
In this lecture, we see ...

- Having seen a few examples of the real-space RG transformations, we formulate it as a general framework for discussing the phase diagram and the critical phenomena.
- As an exactly-treatable example of the RG framework, we consider the Gaussian model, which is easy to solve and provides us the starting point for perturbative renormalization group.

[6-1] Fixed-point and scaling operators

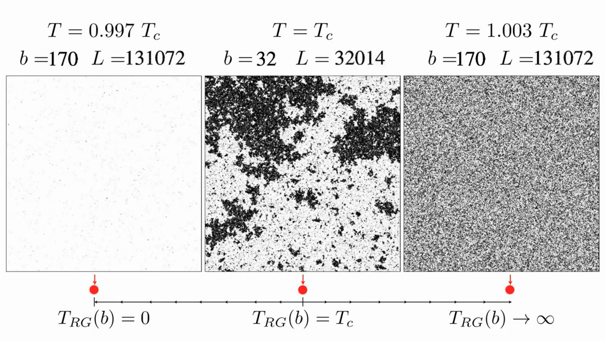
- As a Gedunkenexperiment, we consider a generic Hamiltonian, and its exact renormalization group transformation. (As long as it exists, it doesn't matter whether or not we can actually compute such things.)
- We'll see that the RGT defines a "RG-flow" in the parameter space, which provides us with a framework of understanding the phase diagram.

Critical point is scale-invariant



“https://youtu.be/fi-g2ET97W8” by Douglas Ashton

RG flow



"https://youtu.be/MxRddFrEnPc" by Douglas Ashton

Generic Hamiltonian

- Any Hamiltonian is expressed as an expansion w.r.t. local operators.

$$\mathcal{H}_a(S|\mathbf{K}, L) = -\sum_{\mathbf{x}} \sum_{\alpha} K_\alpha S_\alpha(\mathbf{x}) \quad (1)$$

where $\{S_\alpha\}$ spans the space of all local operators, i.e.,

$$\forall Q(\mathbf{x}) \; \exists q_\alpha \; \left(Q(\mathbf{x}) = \sum_{\alpha} q_\alpha S_\alpha(\mathbf{x}) \right) \quad (2)$$

- Example: A generic model defined with Ising spins.

$$\begin{array}{ll}
K_1 = H & S_1(\mathbf{x}) = S_{\mathbf{x}} \\
K_2 = J_x & S_2(\mathbf{x}) = S_{\mathbf{x}}\, S_{\mathbf{x}+\mathbf{a}_x} \\
K_3 = J_y & S_3(\mathbf{x}) = S_{\mathbf{x}}\, S_{\mathbf{x}+\mathbf{a}_y} \\
K_4 = Q & S_4(\mathbf{x}) = S_{\mathbf{x}}\, S_{\mathbf{x}+\mathbf{a}_x}\, S_{\mathbf{x}+2\mathbf{a}_x} \\
K_5 = Q & S_5(\mathbf{x}) = S_{\mathbf{x}}\, S_{\mathbf{x}+\mathbf{a}_x}\, S_{\mathbf{x}+\mathbf{a}_y} \\
\vdots & \vdots
\end{array}$$

($\mathbf{a}_x, \mathbf{a}_y, \cdots$: lattice unit vectors)

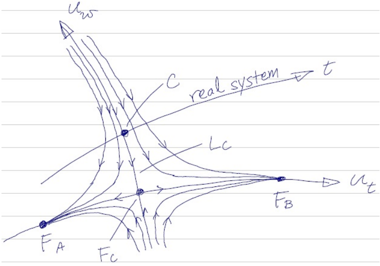
RG flow diagram

- The RGT

$$\mathcal{H}_a(\phi, \mathbf{K}) \to \mathcal{H}_a(\acute{\phi}, \acute{\mathbf{K}})$$

can be regarded as a map from the parameter space onto itself

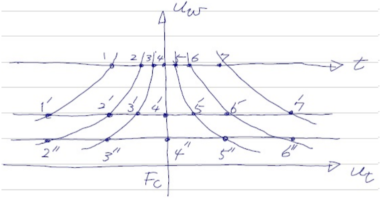
$$\mathbf{K} \to \acute{\mathbf{K}} \equiv \mathcal{R}_b \mathbf{K}$$



- An RG trajectory is a RGT-invariant curve.
- We assume that the trajectory is continuous. (In other words, the RGT is defined for continuous b, such that $\mathcal{R}_{b_1}\mathcal{R}_{b_2} = \mathcal{R}_{b_1 b_2}$.)
- A trajectry converging to the unstable fixed point ($\mathrm{F_C}$) is called a critical line ($\mathrm{L_C}$). The parameter along it is called irrelevant (u_w).
- The parameter along a trajectry emanating from the unstable fixed point is called relevant. (u_t).

Critical properties are controled by unstable fixed-point

- RGT with b maps the points $1, 2, \cdots, 7$ to $1', 2', \cdots, 7'$.
- RGT with $b' > b$ maps the narrower region including only $2, 3, 4, 5, 6$, to $2'', 3'', \cdots, 6''$, distributed in the same range of u_t, but closer to the $u_w = 0$ line.



- In this way, a narrower region is mapped closer to the $u_w = 0$ line. So, the critical properties on the t-axis, is identical to the property around the unstable fixed point ("$\mathrm{F_C}$") on the $u_w = 0$ line.
- The irrelevant fields of our system determine how far we must approach to the critical point to observe the correct critical behavior.
- Applying a small irrelevant field does not qualitatively change the nature of the critical point, while a relevant field does.

Expansion around unstable fixed point

- Consider the local Hamiltonian $\mathcal{H}_a(\mathbf{S}(\mathbf{x}), \mathbf{x})$ and its fixed point form:

$$\mathcal{H}_a^*(\mathbf{S}(\mathbf{x}), \mathbf{x}) \equiv \mathcal{H}_a(\mathbf{S}(\mathbf{x}), \mathbf{x}|\mathbf{K}^*). \tag{3}$$

(In what follows, we drop some or all of the parameters, a, $\mathbf{x}$ and $\mathbf{S}(\mathbf{x})$, and use the abbreviation like $\mathcal{H}^*$ for $\mathcal{H}_a^*(\mathbf{S}(\mathbf{x}), \mathbf{x})$.)

- Let us denote the RGT by $\mathcal{R}_b$ where b is the renormalization factor. Then, $\mathcal{R}_b(\mathcal{H}^*) = \mathcal{H}^*$.
- Let us expand the Hamiltonian around this fixed point.

$$\mathcal{H} = \mathcal{H}^* - \sum_\alpha h_\alpha S_\alpha(\mathbf{x}) = \mathcal{H}^* - \mathbf{h} \cdot \mathbf{S} \tag{4}$$

where h_α is the deviation of the parameter K_α from its fixed-point value, i.e., $h_\alpha \equiv K_\alpha - K_\alpha^*$

Linearization of RGT

- Now consider the transformation applied to the local Hamiltonian near the fixed point:

$$\mathcal{R}_b(\mathcal{H}^* - \mathbf{h}\cdot\mathbf{S}(\mathbf{x})) = \mathcal{H}^* - \acute{\mathbf{h}}\cdot\mathbf{S}(\mathbf{x})$$

- To the lowest order, $\acute{\mathbf{h}}$ depends linearly on $\mathbf{h}$ in the lowest order, i.e., a linear operator T_b exists such that

$$\acute{\mathbf{h}} \approx T_b\mathbf{h}.$$

- We assume that T_b is diagonalizable with real eigenvalues.

$$P^{-1}T_bP = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix} \equiv \Lambda_b$$

Scaling fields and scaling operators

- By defining

$$\mathbf{u} \equiv P^{-1}\mathbf{h}, \text{ and } \phi \equiv P^{\mathsf{T}}\mathbf{S}$$

we obtain

$$\mathbf{u}\cdot\phi = (P^{-1}\mathbf{h})^{\mathsf{T}}(P^{\mathsf{T}}\mathbf{S}) = \mathbf{h}^{\mathsf{T}}(P^{-1})^{\mathsf{T}}P^{\mathsf{T}}\mathbf{S} = \mathbf{h}\cdot\mathbf{S}.$$

In addition, $\mathbf{u}$ transforms as

$$\acute{\mathbf{u}} \equiv \mathcal{R}_b\mathbf{u} = P^{-1}\acute{\mathbf{h}} = P^{-1}T_b\mathbf{h} = P^{-1}T_bP\mathbf{u} = \Lambda_b\mathbf{u},$$

namely, $\acute{u}_\mu = b^{y_\mu}u_\mu$ with $y_\mu \equiv \log_b \lambda_\mu$.

$u_\mu =$ "scaling field", $\phi_\mu =$ "scaling operator",

$y_\mu =$ "scaling eigenvalue" $\left(\begin{array}{ll} y_\mu > 0 & \rightarrow u_\mu \text{ is relevant} \\ y_\mu < 0 & \rightarrow u_\mu \text{ is irrelevant} \end{array}\right)$

Scaling dimensions

- We have seen that we can formulate the RGT for a general Hamiltonian expanded around a fixed point

$$\mathcal{H}(\phi) = \mathcal{H}^*(\phi) - \mathbf{u}\cdot\phi,$$

as

$$\acute{\mathcal{H}}(\acute{\phi}) \equiv \mathcal{R}_b\mathcal{H}(\phi) = \mathcal{H}^*(\acute{\phi}) - \sum_{\mu} b^{y_\mu} u_\mu \acute{\phi}_\mu.$$

- The scaling property of ϕ_μ is determined by y_μ through the condition

$$\int d\mathbf{x} u_\mu(\mathbf{x})\phi_\mu(\mathbf{x}) = \int d\acute{\mathbf{x}}\acute{u}_\mu(\acute{\mathbf{x}})\acute{\phi}_\mu(\acute{\mathbf{x}}) + (\text{short length-scale term})$$

with $\acute{\mathbf{x}} = b^{-1}\mathbf{x}$ and $\acute{u}_\mu = b^{y_\mu}u_\mu$. Namely,

$$\acute{\phi}_\mu(\acute{\mathbf{x}}) \approx b^{x_\mu}\phi_\mu(\mathbf{x}) \quad \text{with} \quad x_\mu = d - y_\mu$$

which is called "scaling dimension" of the scaling operator ϕ_μ.

Scaling form of correlation functions

- For correlation function in the long-length scale, we have

$$G_\mu(|\acute{\mathbf{x}} - \acute{\mathbf{y}}|, \acute{\mathbf{K}}) = \langle \acute{\phi}_\mu(\acute{\mathbf{x}}) \acute{\phi}_\mu(\acute{\mathbf{y}}) \rangle_{\mathcal{H}(\acute{\phi}, \acute{\mathbf{K}})}$$
$$\approx b^{2x_\mu} \langle \phi_\mu(\mathbf{x}) \phi_\mu(\mathbf{y}) \rangle_{\mathcal{H}(\phi, \mathbf{K})} = b^{2x_\mu} G_\mu(|\mathbf{x} - \mathbf{y}|, \mathbf{K}),$$

or $$G_\mu(r, \mathbf{K}) \approx \frac{1}{b^{2x_\mu}} G_\mu\left(\frac{r}{b}, \acute{\mathbf{K}}\right)$$

- Let us consider the case where b is large enough that all irrelevant field in $\acute{\mathbf{K}}$ are regarded as zero.
- When we have only one non-zero relevant field, say t,

$$G_\mu(r, t) \approx \frac{1}{b^{2x_\mu}} G_\mu\left(\frac{r}{b}, b^{y_t} t\right).$$

By choosing $b = r$, we obtain

$$G_\mu(r, t) \approx \frac{1}{r^{2x_\mu}} g_\mu\left(\frac{r}{t^{-1/y_t}}\right) \quad (g_\mu(x) \equiv G_\mu(1, x^{y_t}))$$

Critical exponents ν and η

- Let us consider what we can deduce from the scaling form

$$G_\mu(r,t) \approx \frac{1}{r^{2x_\mu}} g_\mu\left(\frac{r}{t^{-1/y_t}}\right).$$

- First, by comparing it with the defining equation of the correlation length, $G_\mu(r,t) \propto r^{-\omega} e^{-r/\xi(t)}$ we can derive

$$\xi(t) \propto t^{-\frac{1}{y_t}} \quad \Rightarrow \quad \nu = \frac{1}{y_t}.$$

- Second, by taking the limit $t \to 0$,

$$G_\mu(r, t=0) \approx \frac{1}{r^{2x_\mu}} g_\mu(0) \tag{5}$$

which means

$$d - 2 + \eta_\mu = 2x_\mu$$

Order parameters and critical exponent β

- Consider the expectation value of a scaling field ϕ_μ

$$m_\mu(\mathbf{u}) \equiv \langle \phi_\mu(\mathbf{x}) \rangle_\mathbf{u} \approx \langle b^{-x_\mu} \acute{\phi}_\mu(\acute{\mathbf{x}}) \rangle_\mathbf{u} = b^{-x_\mu} m_\mu(\acute{\mathbf{u}}).$$

  It follows that $m_\mu(\mathbf{0}) = 0$ if $x_\mu \neq 0$, which we assume below.
- Suppose that spontaneous "magnetization" exists (i.e., $\langle \phi_\mu \rangle > 0$) slightly away from the critical point. When we have only one non-zero relevant field $t \equiv u_\nu$,

$$m_\mu(t) \approx b^{-x_\mu} m_\mu(b^{y_t} t).$$

- By choosing $b = (t/t_0)^{-1/y_t}$, with t_0 being any constant, we obtain

$$m_\mu(t) \propto t^{\frac{x_\mu}{y_t}},$$

  Thus, the critical exponent β is related to the scaling dimensions, i.e.,

$$\beta = \frac{x_\mu}{y_t}.$$

[6-2] Gaussian model and Gaussian fixed point

- Consider the Gaussian model:

$$\mathcal{H}_a(\phi|\rho,t) \equiv \int_a^L d^d\mathbf{x}\,\left(\rho(\nabla\phi_{\mathbf{x}})^2 + t\phi_{\mathbf{x}}^2 - h\phi_{\mathbf{x}}\right)$$
$$= \int_{\pi/L}^{\pi/a} \frac{d^d\mathbf{k}}{(2\pi)^d}(\rho k^2 + t)\phi_{\mathbf{k}}^2 - h\phi_{\mathbf{0}}.$$

(∗ The lower-bound of the integrals symbolically specifies the short-range cutoff.)

- We will apply the RG transformation:

$$\text{Partial Trace:}\quad \mathcal{H}_a(\phi|\rho,t,h) \to \mathcal{H}_{ba}(\tilde{\phi}|\tilde{\rho},\tilde{t},\tilde{h})$$
$$\text{Rescaling:}\quad \mathcal{H}_{ba}(\tilde{\phi}|\tilde{\rho},\tilde{t},\tilde{h}) \to \mathcal{H}_a(\acute{\phi}|\acute{\rho},\acute{t},\acute{h})$$
$$\left(\phi'_{\mathbf{k}'} = b^{-y}\tilde{\phi}_{\mathbf{k}} \quad (\mathbf{k}' \equiv b\mathbf{k})\right)$$

Partial trace of short-range fluctuation

- (Partial trace) $\mathcal{H}_a(\phi|\rho, t, h) \to \mathcal{H}_{ab}(\tilde{\phi}|\tilde{\rho}, \tilde{t}, \tilde{h})$

  Since each wave-number component is independent from the others, the summation over $\phi_{\mathbf{k}}$ for $|\mathbf{k}| > \pi/2a$ results simply in a multiplicative constant:

$$e^{-\mathcal{H}_{ba}(\tilde{\phi}|\tilde{\rho},\tilde{t},\tilde{h})} \equiv \int d\{\phi_{\mathbf{k}}\}_{|\mathbf{k}|>\frac{\pi}{ba}} e^{-\int_{\pi/L}^{\pi/a} \frac{d^d\mathbf{k}}{(2\pi)^d}\left(\rho k^2+t\right)\phi_{\mathbf{k}}^2+h\phi_{\mathbf{0}}}$$

$$\sim e^{-\int_{\pi/L}^{\pi/ba} \frac{d^d\mathbf{k}}{(2\pi)^d}\left(\rho k^2+t\right)\phi_{\mathbf{k}}^2+h\phi_{\mathbf{0}}},$$

or $$\mathcal{H}_{ba}(\tilde{\phi}|\tilde{\rho}, \tilde{t}) = \int_{\pi/L}^{\pi/ba} \frac{d^d\mathbf{k}}{(2\pi)^d} \left(\rho k^2 + t\right) \phi_{\mathbf{k}}^2 - h\phi_{\mathbf{0}}.$$

  In short, the partial trace amounts to

$$\tilde{\phi}_{\mathbf{k}} = \phi_{\mathbf{k}} \quad \left(\text{for } |\mathbf{k}| < \frac{\pi}{ba}\right), \quad (\tilde{\rho}, \tilde{t}, \tilde{h}) = (\rho, t, h).$$

Rescaling

- (Rescaling) $\mathcal{H}_{ba}(\tilde{\phi}|\tilde{\rho},\tilde{t}) \to \mathcal{H}_a(\acute{\phi}|\acute{\rho},\acute{t})$ $\left(\acute{\phi}_{\acute{\mathbf{k}}} = b^{-y_h}\tilde{\phi}_{\mathbf{k}}\ (\acute{\mathbf{k}} \equiv b\mathbf{k})\right)$

$$\begin{aligned}\mathcal{H}_a(\acute{\phi}|\acute{\rho},\acute{t},\acute{h}) &= \int_{b\pi/L}^{\pi/a} \frac{d^d\acute{\mathbf{k}}}{(2\pi)^d} b^{-d}\left(\rho b^{-2}k'^2 + t\right) b^{2y_h}\acute{\phi}^2_{\acute{\mathbf{k}}} - h\phi_{\mathbf{0}} \\ &= \int_{b\pi/L}^{\pi/a} \frac{d^d\acute{\mathbf{k}}}{(2\pi)^d} b^{-(d+2)+2y_h}\left(\rho\acute{\mathbf{k}}^2 + b^2 t\right)\acute{\phi}^2_{\acute{\mathbf{k}}} - b^{y_h}h\acute{\phi}_{\mathbf{0}}\end{aligned}$$

The exponent y_h should be chosen so that ρ is unchanged by the RG transformation. Namely, $y_h = (d+2)/2$.
Then,

$$\mathcal{H}_a(\acute{\phi}|\acute{\rho},\acute{t},\acute{h}) = \int_{b\pi/L}^{\pi/a} \frac{d^d\acute{\mathbf{k}}}{(2\pi)^d}\left(\rho\acute{k}^2 + \acute{t}\right)\acute{\phi}^2_{\acute{\mathbf{k}}} - \acute{h}\acute{\phi}_{\mathbf{0}}$$

with $\acute{t} \equiv b^2 t$, and $\acute{h} \equiv b^{y_h}h$.

RG transformation of the Gaussian model

To summarize,

- By RG transformation,

$$\mathcal{H}_a(\phi|\rho, t, h) = \int_{b\pi/L}^{\pi/a} \frac{d^d\mathbf{k}}{(2\pi)^d} \left(\rho k^2 + t\right) \phi_{\acute{\mathbf{k}}}^2 - h\phi_{\mathbf{0}}$$

is transformed into

$$\mathcal{H}_a(\acute{\phi}|\acute{\rho}, \acute{t}, \acute{h}) = \int_{b\pi/L}^{\pi/a} \frac{d^d\acute{\mathbf{k}}}{(2\pi)^d} \left(\rho \acute{k}^2 + \acute{t}\right) \acute{\phi}_{\acute{\mathbf{k}}}^2 - \acute{h}\acute{\phi}_{\mathbf{0}}$$

with

$$\acute{\mathbf{k}} = b\mathbf{k}, \quad \acute{\phi}_{\acute{\mathbf{k}}} = b^{-y_h}\tilde{\phi}_{\mathbf{k}}, \quad \acute{t} = b^{y_t}\tilde{t}, \quad \acute{h} = b^{y_h}h \tag{6}$$

with

$$y_t \equiv 2 \quad \text{and} \quad y_h \equiv \frac{d+2}{2}. \tag{7}$$

RGT on $\phi_{\mathbf{x}}$

- While in [6-1] we saw $x_\mu = d - y_\mu$ in general, its direct derivation in the case of Gaussian model clarifies the meaning of RGT.
- Considering the Fourier components of $\acute{\phi}_{\acute{\mathbf{x}}}$,

$$\acute{\phi}_{\acute{\mathbf{x}}} = \acute{L}^{-d} \sum_{\acute{k}}^{\pi/a} e^{i\acute{\mathbf{k}}\acute{\mathbf{x}}} \acute{\phi}_{\acute{\mathbf{k}}} = b^d L^{-d} \sum_{\mathbf{k}}^{\pi/ab} e^{i\mathbf{k}\mathbf{x}} b^{-y} \phi_{\mathbf{k}}$$

$$= b^{d-y} L^{-d} \sum_{\mathbf{k}}^{\pi/ab} e^{i\mathbf{k}\mathbf{x}} \phi_{\mathbf{k}} = b^x [\phi_{\mathbf{x}}]_{k<\frac{\pi}{ab}}$$

- Here, $[\phi_{\mathbf{x}}]_{k<k^*} \equiv L^{-d} \sum_{\mathbf{k}}^{k^*} e^{i\mathbf{k}\mathbf{x}} \phi_{\mathbf{k}}$ is something one obtains after filtering out the short wave-length part ($k > k^*$) from $\phi_{\mathbf{x}}$. Therefore, $\phi_{\mathbf{x}}$ and $[\phi_{\mathbf{x}}]_{k<k^*}$ are identical in the renormalized description.

Implication of RGT

- In [6-1], we saw, in general,

$$\nu = \frac{1}{y_t}$$
$$d - 2 + \eta_\mu = 2x_\mu$$

- For the Gaussian model, we have derived

$$y_t = 2 \quad \text{and} \quad y_h = \frac{d+2}{2}$$

- Therefore, for the gaussian model

$$\nu = \frac{1}{2} \quad \text{and} \quad \eta = 0.$$

Homework (Submit your report on one of the following)

- By an argument similar to the one resulting in $\beta_\mu = x_\mu / y_t$, show that the critical exponent γ_μ that describes the temperature-dependence of the susceptibility, $\chi_\mu \equiv \partial \langle \phi_\mu(\mathbf{x}) \rangle / \partial u_\mu \propto t^{-\gamma_\mu}$, is related to the scaling dimensions/eigenvalues as $\gamma_\mu = \dfrac{y_\mu - x_\mu}{y_t} = \dfrac{2y_\mu - d}{y_t}$.
- Consider a system for which the susceptibility χ_μ diverges as one approaches the critical point keeping the condition $u_\mu = 0$. Does application of infinitesimal field u_μ qualitatively change the critical properties? Can we say the opposite, i.e., that the field does not essentially change the nature of the transition whenever $\chi_\mu < \infty$?
- In the rescaling of the Gaussian model, we fixed y_h so that the ρ would not change. In principle, we should be able to obtain some RGT by fixing other parameters in stead of ρ. What would we have obtained, for example, if we had fixed t rather than ρ?