Lecture 10: ϵ-expansion and Wilson-Fisher fixed point

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In this lecture, we see ...

- By applying the perturbative RG to GFP, we will find a new fixed point near the GFP. (Wilson-Fisher fixed point (WFFP))
- By replacing the GFP and the WFFP by their multi-component counterparts, we can obtain the ϵ-expansion of the universality classes of the XY model ($n = 2$) and of the Heisenberg model ($n = 3$).

[10-1] Wilson-Fisher fixed point

- By inspecting the RG flow equation around GFP, we can obtain an $\epsilon(\equiv 4 - d)$ dependent fixed point and its scaling properties to the first order in ϵ (ϵ-expansion).
- From this result one can obtain the lowest order approximation to the Wilson-Fisher fixed point, which is suppose to (exactly) describe the Ising universality class in dimensions $2 < d < 4$.

RG flow equation around GFP

- Now, we are ready to actually compute the RG flow around the GFP searching for a new fixed point for the ϕ^4 model.
- Our tool is the RG flow equation around a fixed point.

$$\frac{dg_n}{d\lambda} = y_n g_n - \sum_{lm} c^n_{lm} g_l g_m + O(g^3) \quad (\lambda \equiv \log b) \tag{1}$$

- For the GFP, we already know

$$\phi_n \equiv [\![\phi^n]\!], \quad y_n = d - x_n, \quad x_n = nx = \frac{n}{2}(d-2)$$
$$c^n_{lm} \equiv \binom{l}{k}\binom{m}{k} k! \quad \left(k \equiv \frac{l+m-n}{2}\right) \tag{2}$$

The Z_2 symmetry

- Let us focus on the relevant fields at the GFP:

$$h \equiv g_1, \quad t \equiv g_2, \quad v \equiv g_3, \quad u \equiv g_4$$

- Note that (1) and (2) ensures that when we start with even fields only, odd fields are not generated by the RGT.
- In addition, we know that the critical point of the Ising model possesses the symmetry with respect to $S \leftrightarrow -S$.
- Therefore, we expect that the fixed point representing the Ising criticality should be found in the "even parity" manifold, i.e., $h = v = 0$.

ϵ-expansion

- In terms of the remaining fields, t and u, the flow equations are

$$\frac{dt}{d\lambda} = y_t t - c^t_{tt} t^2 - 2c^t_{tu} tu - c^t_{uu} u^2 + O(g^3) \quad (3)$$

$$\frac{du}{d\lambda} = y_u u - c^u_{tt} t^2 - 2c^u_{tu} tu - c^u_{uu} u^2 + O(g^3) \quad (4)$$

with $y_t = 2$ and $y_u = 4 - d \equiv \epsilon$.

- Hereafter, we regard ϵ as a small quantity.

- Let (t^*, u^*) be the non-trivial solution to the fixed-point equation, i.e., they are not zero and make the RHSs of (3) and (4) zero.

- By considering the order in ϵ, we see $t^* = O(\epsilon^2)$ and $u^* = O(\epsilon)$.
($\because$ By perturbation assuption, both u^* and t^* are small. Then, in (3), the only term that can possibly be the same order as t is u^2. Therefore, $t^* \sim {u^*}^2$. With this in mind, inspecting (4) we see that ϵu must be comparable to u^2, so $u^* \sim O(\epsilon)$.)

Wilson-Fisher fixed point

- Now, only keeping the terms that can make difference, we obtain

$$\frac{dt}{d\lambda} = 2t - 96u^2 - 24tu \quad (\equiv A) \tag{5}$$

$$\frac{du}{d\lambda} = \epsilon u - 72u^2 - 16tu \quad (\equiv B) \tag{6}$$

$$\left(c^t_{uu} = \binom{4}{3}\binom{4}{3} 3! = 96, \quad c^u_{uu} = \binom{4}{2}\binom{4}{2} 2! = 72, \quad \text{etc.} \right)$$

- Then, the fixed point is

$$(t^*, u^*) = \left(\frac{\epsilon^2}{108}, \frac{\epsilon}{72} \right) \tag{7}$$

- We regard this as the lowest order approximation to the new fixed point that we've been seeking for. (Wilson-Fisher fixed point (WFFP))

Linearization around the WFFP

- To obtain the scaling properties of the WFFP, we need to re-expand the series-expansion around the WFFP.

- So, let us define

$$\Delta u \equiv u - u^*$$
$$\Delta t \equiv t - t^*$$

and recast (5) and (6) in the form $\frac{d}{d\lambda}\begin{pmatrix}\Delta t \\ \Delta u\end{pmatrix} = Y\begin{pmatrix}\Delta t \\ \Delta u\end{pmatrix}$.

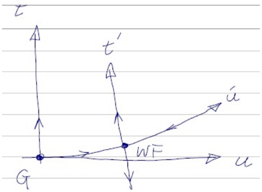
- Obviously, the matrix Y can be obtained as

$$Y \equiv \begin{pmatrix} \frac{\partial A}{\partial t} & \frac{\partial A}{\partial u} \\ \frac{\partial B}{\partial t} & \frac{\partial B}{\partial u} \end{pmatrix}_{\Delta t = \Delta u = 0} = \begin{pmatrix} 2 - 24u^* & -192u^* \\ -16u^* & \epsilon - 144u^* \end{pmatrix}$$
$$= \begin{pmatrix} 2 - \frac{1}{3}\epsilon & -\frac{8}{3}\epsilon \\ -\frac{2}{9}\epsilon & -\epsilon \end{pmatrix}.$$

Scaling properties of the WFFP

- Thus, the linearized RG flow equation around the new fixed point is

$$\frac{d}{d\lambda}\begin{pmatrix}\Delta t \\ \Delta u\end{pmatrix} = \begin{pmatrix} 2-\frac{1}{3}\epsilon & -\frac{8}{3}\epsilon \\ -\frac{2}{9}\epsilon & -\epsilon \end{pmatrix}\begin{pmatrix}\Delta t \\ \Delta u\end{pmatrix}.$$



- Since the off-diagonal elements do not contribute to the eigenvalues to $O(\epsilon)$,

$$y_u^{\mathrm{WF}} = -\epsilon \quad \text{and} \quad y_t^{\mathrm{WF}} = 2 - \frac{\epsilon}{3}$$

- The "t-like" scaling field is relevant.

$$y_t^{\mathrm{3DWF}} \approx 1.666\cdots \qquad \left(y_t^{\mathrm{3DIsing}} \approx 1.59,\right)$$

$$y_t^{\mathrm{2DWF}} \approx 1.333\cdots \qquad \left(y_t^{\mathrm{2DIsing}} = 1\right)$$

Scaling eigenvalue of h at WFFP

- Writing down the RG flow equation for h, which has been neglected so far,

$$\frac{dh}{d\lambda} = y_h h - 2c^h_{th} th + (u^2 h\text{-term}) = \frac{d+2}{2} h - 4th + \cdots$$
$$\approx \left(\frac{d+2}{2} - 4t^* + \cdots \right) h$$

- Therefore, $y_h^{\mathrm{WF}} = \dfrac{d+2}{2} + O(\epsilon^2)$. In other words,

$$\eta^{\mathrm{WF}} = 2x_h^{\mathrm{WF}} - d + 2 = d + 2 - 2y_h^{\mathrm{WF}} = 0 + O(\epsilon^2).$$

This should be compared with

$$\eta^{\mathrm{3dIsing}} = 0.022(3) \quad \text{and} \quad \eta^{\mathrm{2dIsing}} = 0.25$$

Irrelevancy of other operators

- Even if some field is irrelevant at the GFP, it may turn relevant at the WFFP. If so, it alters the final destination of the RG flow, in which case the WFFP is not the contoring FP.
- The RG flow equation for g_n around the GFP is

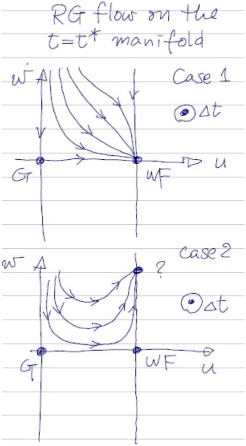
$$\frac{dg_n}{d\lambda} = \left(d - \frac{n}{2}(d-2)\right) g_n - 12n(n-1)ug_n,$$

- Remembering that $u^* = \epsilon/72$,

$$y_n^{\mathrm{WF}} = \left(d - \frac{n}{2}(d-2)\right) - 12n(n-1)\frac{4-d}{72}$$

- For $n \geq 6$, we have negative y_n:

$$y_n^{\mathrm{WF}} = \frac{18 - 2n - n^2}{6}\ (d=3), \frac{6 + n - n^2}{3}\ (d=2).$$



[10-2] $O(n)$ models

- To apply the perturbative RG to the XY (O(2)) and the Heisenberg (O(3)) models we will introduce the multi-component ϕ^4 model.
- We can then construct the RG flow equation as before.

Multi-component ϕ^4 model

- Let us apply the perturbative RG to the XY (O(2)) or the Heisenberg (O(3)) models.
- To follow the same line of argument as before, we need something analogous to the ϕ^4 model to start with.
- So, let us consider multi-component field

$$\boldsymbol{\phi}(\mathbf{x}) \equiv \left(\phi^1(\mathbf{x}), \phi^2(\mathbf{x}), \cdots, \phi^n(\mathbf{x})\right)^{\mathsf{T}}$$

  and the multi-component ϕ^4 model:

$$\mathcal{H} \equiv \int d\mathbf{x} \left(|\nabla \boldsymbol{\phi}|^2 + t\boldsymbol{\phi}^2 + u(\boldsymbol{\phi}^2)^2 - h\phi^1\right)$$

- If $t = u = h = 0$, the n-components are independent and each represents a Gaussian fixed point. Therefore, it is a fixed point for the new Hamiltonian. (We call this fixed point the GFP, too.)

Correlation functions

- To get familiarized with the new model, let us consider $\langle \phi^2(\mathbf{x})\phi^2(\mathbf{y})\rangle_{\mathrm{GFP}}$.
- Since we can use Wick's theorem for the multi-component GFP,

$$\begin{aligned}
&\langle \phi^2(\mathbf{x})\phi^2(\mathbf{y})\rangle \\
&\quad = \langle \phi^\alpha(\mathbf{x})\phi^\alpha(\mathbf{x})\phi^\beta(\mathbf{y})\phi^\beta(\mathbf{y})\rangle \quad \text{(Einstein's convention)} \\
&\quad = \langle \phi^\alpha(\mathbf{x})\phi^\alpha(\mathbf{x})\rangle\langle \phi^\beta(\mathbf{y})\phi^\beta(\mathbf{y})\rangle \\
&\qquad + 2\langle \phi^\alpha(\mathbf{x})\phi^\beta(\mathbf{y})\rangle\langle \phi^\alpha(\mathbf{x})\phi^\beta(\mathbf{y})\rangle \\
&\quad = n^2 G^2(0) + 2nG^2(r)
\end{aligned}$$

where $r \equiv |\mathbf{x} - \mathbf{y}|$ and $G(r) \equiv \langle \phi^1(\mathbf{x})\phi^1(\mathbf{y})\rangle \approx r^{-2x}$ as usual.

Diagramatic representation

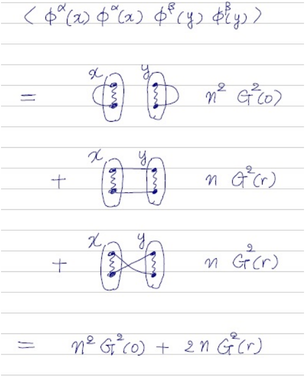
- We have seen that

$$\langle \phi^2(\mathbf{x})\phi^2(\mathbf{y})\rangle = n^2 G^2(0) + 2nG^2(r)$$

- Compared with the previous case of $n = 1$, the difference is the factors n^2 and n.
- For a given pattern of Wick paring, draw the diagram like the one in the right:

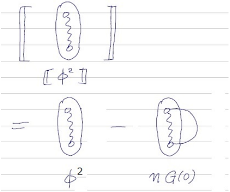
wavy lines	$\leftrightarrow$	repeated indices
regular lines	$\leftrightarrow$	Wick paring

- To the term represented by a diagram with g loops, we assign the factor n^g.



Scaling operator ϕ_2

- As before, we can define the normal-order product, $[\![\cdots]\!]$, as the operator that we obtain after removing all contributions from the diagrams with inner connections.



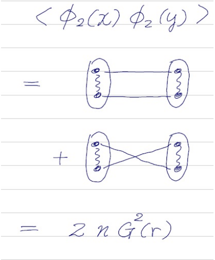
- For example,

$$\phi_2 \equiv \left[\!\!\left[\phi^2 \right]\!\!\right] = \phi^2 - nG^2(0)$$

- For the correlator of two ϕ_2s, we have

$$\langle \phi_2(\mathbf{x}) \phi_2(\mathbf{y}) \rangle = 2nG^2(r)$$

(See the diagram on the right.)



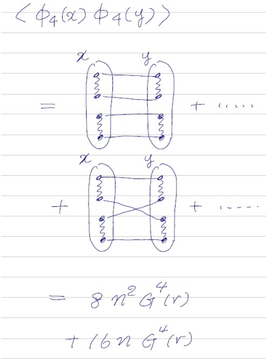
Scaling operator ϕ_4

- Similarly, we define ϕ_4 as

$$\phi_4(\mathbf{x}) \equiv \left[\!\left[\left(\phi^2(\mathbf{x})\right)^2 \right]\!\right]$$

- Then, the correlator becomes

$$\begin{aligned}
&\langle \phi_4(\mathbf{x}) \phi_4(\mathbf{y}) \rangle \\
&\quad = (\text{Two-loop terms}) \\
&\quad\quad + (\text{One-loop terms}) \\
&\quad = 8n^2 G^4(r) + 16nG^4(r) \\
&\quad = (8n^2 + 16n) G^4(r)
\end{aligned}$$



$c^u_{tt}, c^t_{tt}, c^u_{tu}, c^t_{tu}$ for $O(n)$ GFP

- First, let us expand $\phi_2(\mathbf{x})\phi_2(\mathbf{y})$.

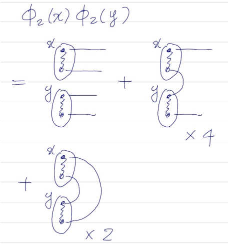
$$\phi_2(\mathbf{x})\phi_2(\mathbf{y}) \approx \phi_4(\mathbf{x}) + 4G(r)\phi_2(\mathbf{x}) + \cdots .$$

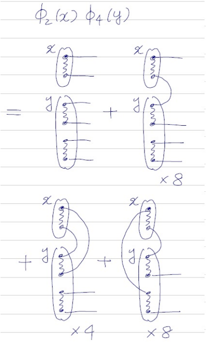
Thus, we obtain $c^u_{tt} = 1$ and $c^t_{tt} = 4$.

- For $\phi_2(\mathbf{x})\phi_4(\mathbf{y})$, we obtain

$$\begin{aligned}\phi_2(\mathbf{x})\phi_4(\mathbf{y}) &= \phi_6(\mathbf{x}) + 8G(r)\phi_4(\mathbf{x}) \\ &\quad + 4nG^2(r)\phi_2(\mathbf{x}) + 8G^2(r)\phi_2(\mathbf{x}) \\ &= \phi_6 + 8G\phi_4 + (4n+8)G^2\phi_2 + \cdots .\end{aligned}$$

We obtain $c^u_{tu} = 8$ and $c^t_{tu} = 4(n+2)$.





Wilson-Fisher FP for $O(n)$ GFP

- The RG flow equation is

$$\begin{cases} \frac{dt}{d\lambda} & = 2t - 32(n+2)u^2 - 8(n+2)tu & \equiv A \\ \frac{du}{d\lambda} & = \epsilon u - 8(n+8)u^2 - 16tu & \equiv B \end{cases}$$

$$\Rightarrow \ (t^*, u^*) = \left(\frac{\epsilon^2}{4(n+8)^2}, \frac{\epsilon}{8(n+8)} \right)$$

- The flow equation for t around WFFP is

$$\frac{dt}{d\lambda} = (2 - 8(n+2)u^*)t \quad \Rightarrow \quad y_t^{\mathrm{WF}} = 2 - \frac{n+2}{n+8}\epsilon$$

- For h, we have

$$\frac{dh}{d\lambda} = (y_h^{\mathrm{G}} + O(\epsilon^2))h = \frac{d+2}{2}h$$

$$\Rightarrow \quad y_h^{\mathrm{WF}} = \frac{d+2}{2} = 3 - \frac{\epsilon}{2}$$

ϵ-expansion summary

		Ising ($n=1$)		XY ($n=2$)		Heisenberg ($n=3$)	
		ϵ-exp.	true	ϵ-exp.	true	ϵ-exp.	true
4D	y_t	2	2	2	2	2	2
	y_h	3	3	3	3	3	3
3D	y_t	1.67	1.59	1.60	1.49	1.55	1.41
	y_h	2.5	2.48	2.5	2.48	2.5	2.49
2D	y_t	1.33	1	1.20	—	1.09	—
	y_h	2.0	1.875	2.0	—	2.0	—

Summary

- By applying the perturbative RG to GFP, we have found a new fixed point near the GFP. (Wilson-Fisher fixed point (WFFP))
- We can apply the same perturbative argument to the n-component field ϕ, resulting in the ϵ-expansion of the universality classes of the XY model ($n = 2$) and of the Heisenberg model ($n = 3$). In 3D, the estimates of scaling dimesnions were surprisingly good, whereas even in 2D, they are not so far from the correct values.

Homework

- Obtain the OPE of $\phi_u(\mathbf{x})\phi_u(\mathbf{y})$ at the GFP, and show that

$$c_{uu}^{u} = 8(n+8) \quad \text{and} \quad c_{uu}^{t} = 32(n+2)$$