

# Lecture 8: Operator Product Expansion

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## In this lecture, we see ...

- Product of two scaling operators can be expanded in terms of scaling operators. (OPE)
- Such an expansion determines the RG-flow structure around the fixed point (will be discussed in the next lecture).
- For the Gaussian fixed point, all the coefficients of the OPE can be exactly obtained.

## [8-1] General Framework

- Product of two scaling operators, defined at some distance from each other, can be expanded as a linear combination of scaling operators.
- Considering the 3-point correlators and taking into account of the scaling properties of operators, the general form of the OPE coefficients can be fixed up to universal constants.

## Product of two is expandable

- Previously, we introduced scaling operators  $\{\phi_\mu\}$  as something that spans the space of local operators:

$$\forall Q(\mathbf{x}) \exists q_\mu \left( Q(\mathbf{x}) = \sum_\mu q_\mu \phi_\mu(\mathbf{x}) \right) \quad (1)$$

- At the fixed point, we required that the RGT acts on  $\phi_\mu$  as

$$\mathcal{R}_b \phi_\mu(\mathbf{x}) = b^{x_\mu} \phi_\mu(\mathbf{x})$$

- Let us consider the product of two scaling operators,  $\phi_\mu(\mathbf{x})\phi_\nu(\mathbf{y})$ . This product must appear to be a “local” operator when we view it from a point  $\mathbf{z}$  far away from  $\mathbf{x}$  and  $\mathbf{y}$  (i.e.,  $|\mathbf{z} - \mathbf{x}| \gg |\mathbf{y} - \mathbf{x}|$ ).
- Then, we should be able to expand it:

$$\phi_\mu(\mathbf{x})\phi_\nu(\mathbf{y}) = \sum_\alpha C_{\mu\nu}^\alpha(\mathbf{x} - \mathbf{y}) \phi_\alpha\left(\frac{\mathbf{x} + \mathbf{y}}{2}\right)$$

(Equality holds only when viewed from a distant point.)

## Three-point correlator (1/2)

- Let us consider three-point correlation function:

$$G_{\mu\nu\lambda}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \equiv \langle \phi_\mu(\mathbf{x}) \phi_\nu(\mathbf{y}) \phi_\lambda(\mathbf{z}) \rangle$$

- By applying the RGT and then expanding  $\phi_\mu \phi_\nu$ , we have

$$\begin{aligned} G_{\mu\nu\lambda}(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}) &= b^{x_\mu + x_\nu + x_\lambda} G_{\mu\nu\lambda}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \\ &= b^{x_\lambda + x_\mu + x_\nu} \sum_{\alpha} C_{\mu\nu}^{\alpha}(\mathbf{x} - \mathbf{y}) G_{\alpha\lambda} \left( \frac{\mathbf{x} + \mathbf{y}}{2}, \mathbf{z} \right) \end{aligned} \quad (2)$$

- By reversing the order of the operations, we have

$$\begin{aligned} G_{\mu\nu\lambda}(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}) &= \sum_{\alpha} C_{\mu\nu}^{\alpha}(\hat{\mathbf{x}} - \hat{\mathbf{y}}) G_{\alpha\lambda} \left( \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}}}{2}, \hat{\mathbf{z}} \right) \\ &= \sum_{\alpha} C_{\mu\nu}^{\alpha}(\hat{\mathbf{x}} - \hat{\mathbf{y}}) b^{x_\alpha + x_\lambda} G_{\alpha\lambda} \left( \frac{\mathbf{x} + \mathbf{y}}{2}, \mathbf{z} \right) \end{aligned} \quad (3)$$

## Three-point correlator (2/2)

- Comparing (2) and (3), we conclude

$$C_{\mu\nu}^{\alpha}(\mathbf{r}) = \frac{C_{\mu\nu}^{\alpha}\left(\frac{\mathbf{r}}{b}\right)}{b^{x_{\mu}+x_{\nu}-x_{\alpha}}}$$

This leads to

$$\exists c_{\mu\nu}^{\alpha} \left( C_{\mu\nu}^{\alpha}(r) = \frac{c_{\mu\nu}^{\alpha}}{r^{x_{\mu}+x_{\nu}-x_{\lambda}}} \right).$$

Therefore, we have the operator-product expansion:

$$\phi_{\mu}(\mathbf{x})\phi_{\nu}(\mathbf{y}) = \sum_{\alpha} \frac{c_{\mu\nu}^{\alpha}}{r^{x_{\mu}+x_{\nu}-x_{\lambda}}} \phi_{\alpha}(\mathbf{x}) \quad (\text{OPE})$$

# Universality

- By normalizing the scaling operators so that

$$\lim_{|\mathbf{x}-\mathbf{y}|\rightarrow\infty} |\mathbf{x}-\mathbf{y}|^{2x_\mu} \langle \phi_\mu(\mathbf{x}) \phi_\nu(\mathbf{y}) \rangle = 1$$

the OPE coefficient  $c_{\mu\nu}^\alpha$  can be fixed (and become universal quantities).

- We assume that thus fixed OPE coefficients  $c_{\mu\nu}^\alpha$  are universal, and characterizing property of the fixed-point, together with the scaling dimensions,  $x_\mu$ . In other words, they do not depend on the details of the system, but depend only on the symmetry, the space dimension, etc. (This assumption of universality is similar to the assumption of very existence of the fixed-point of the RGT. It is at least supported by several exactly solvable cases.)

## [8-2] OPE at the Gaussian fixed point

- The scaling operators at the Gaussian fixed point can be obtained through the normal order product:  $\phi_n \equiv \llbracket \phi^n \rrbracket$ .
- We can compute exact OPE coefficients of the gaussian fixed point.



## A hint for scaling operators — Wick's theorem

- Consider the operator  $(\phi(\mathbf{x}))^2$  and its correlation function.

$$\begin{aligned}
 \langle \phi^2(\mathbf{x}) \phi^2(\mathbf{y}) \rangle &= \langle \phi(\mathbf{x})_1 \phi(\mathbf{x})_2 \phi(\mathbf{y})_3 \phi(\mathbf{y})_4 \rangle \\
 &= \langle \phi(\mathbf{x})_1 \phi(\mathbf{x})_2 \rangle \langle \phi(\mathbf{y})_3 \phi(\mathbf{y})_4 \rangle \\
 &+ \langle \phi(\mathbf{x})_1 \phi(\mathbf{y})_3 \rangle \langle \phi(\mathbf{x})_2 \phi(\mathbf{y})_4 \rangle \\
 &+ \langle \phi(\mathbf{x})_1 \phi(\mathbf{y})_4 \rangle \langle \phi(\mathbf{x})_2 \phi(\mathbf{y})_3 \rangle \quad (4) \\
 &= G^2(0) + 2G^2(r)
 \end{aligned}$$

where  $r \equiv |\mathbf{x} - \mathbf{y}|$  and

$$G(r) \sim \frac{1}{r^{2x}} \quad \text{where} \quad x \equiv \frac{d-2}{2}$$

$$\begin{aligned}
 \langle \phi(\mathbf{x})_1 \phi(\mathbf{x})_2 \phi(\mathbf{y})_3 \phi(\mathbf{y})_4 \rangle &= \langle \phi(\mathbf{x})_1 \phi(\mathbf{x})_2 \rangle \langle \phi(\mathbf{y})_3 \phi(\mathbf{y})_4 \rangle \\
 &+ \langle \phi(\mathbf{x})_1 \phi(\mathbf{y})_3 \rangle \langle \phi(\mathbf{x})_2 \phi(\mathbf{y})_4 \rangle \\
 &+ \langle \phi(\mathbf{x})_1 \phi(\mathbf{y})_4 \rangle \langle \phi(\mathbf{x})_2 \phi(\mathbf{y})_3 \rangle \\
 &= (G(0))^2 + 2(G(r))^2 \\
 \int_V \phi(\mathbf{x})^2 &= G(0) \quad \int_V \phi(\mathbf{x}) \phi(\mathbf{y}) = G(r) \\
 r &\equiv |\mathbf{x} - \mathbf{y}|
 \end{aligned}$$

# Scaling operators

What are scaling operators at the Gaussian fixed point?

- If the constant term  $G(0)^2$  in

$$\langle \phi^2(\mathbf{x})\phi^2(\mathbf{y}) \rangle = G(0)^2 + 2G(r)^2$$

were absent,  $\phi^2(\mathbf{x})$  would be regarded as a scaling operator.

- This observation leads us to define

$$\phi_2(\mathbf{x}) \equiv \phi(\mathbf{x})^2 - \langle \phi(\mathbf{x})^2 \rangle,$$

which has the two-point correlator

$$\begin{aligned} \langle \phi_2(\mathbf{x})\phi_2(\mathbf{y}) \rangle &= \langle (\phi^2(\mathbf{x}) - G(0))(\phi^2(\mathbf{y}) - G(0)) \rangle \\ &= \langle \phi^2(\mathbf{x})\phi^2(\mathbf{y}) \rangle - G(0)^2 = 2G(r)^2 = \frac{2}{r^{4x}} \end{aligned}$$

- Therefore,  $\phi_2$  is the scaling operator with the dimension  $x_2 \equiv 2x$ .

## Normal-ordered operator

- The key to finding general scaling operators is to eliminate the diagrams with “internal connections” such as the first term in (4).
- Therefore, it would be convenient to introduce a symbol  $[[\cdot \cdot \cdot]]$  as

$$[[A(\mathbf{x})]] \equiv A(\mathbf{x}) - \left( \begin{array}{c} \text{All terms represented by} \\ \text{diagrams with internal} \\ \text{connections} \end{array} \right)$$

The operator thus defined is called “normal-ordered.”

- When considering correlations among normal-ordered operators, by definition, we can forget about the internal lines. Therefore, 2-point correlators do not have constant terms, which makes the normal-ordered operator  $[[\phi^n]]$  a scaling operator. For example,

$$\langle [[\phi^3]]_x [[\phi^3]]_y \rangle = \text{diagram with 3 internal lines} + \text{diagram with 2 internal lines} + \dots = 6 G^3(r)$$

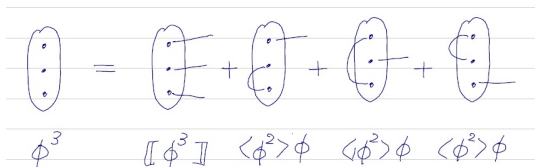
## Scaling operators $\phi_2 \equiv \llbracket \phi^2 \rrbracket$ and $\phi_3 \equiv \llbracket \phi^3 \rrbracket$

- For  $\phi^2$ , as we have seen already

$$\phi_2 \equiv \llbracket \phi^2 \rrbracket = \phi^2 - \langle \phi^2 \rangle$$

- For  $\phi^3$ , from the diagram below, we obtain

$$\phi^3 = \llbracket \phi^3 \rrbracket + 3\langle \phi^2 \rangle \phi$$



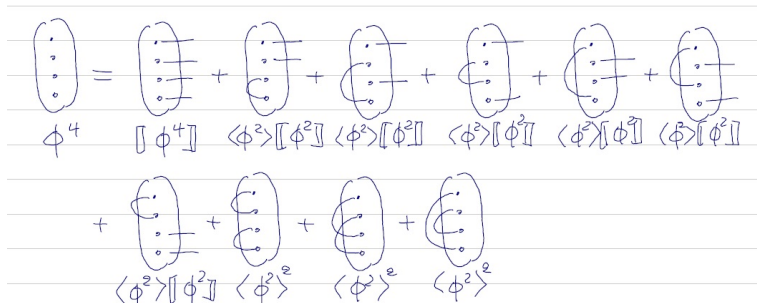
Therefore,

$$\phi_3 \equiv \llbracket \phi^3 \rrbracket = \phi^3 - 3G(0)\phi$$

## Scaling operator $\phi_4 \equiv \llbracket \phi^4 \rrbracket$

- For  $\phi^4$ , again from the diagram, we obtain

$$\phi^4 = \llbracket \phi^4 \rrbracket + 6\langle \phi^2 \rangle \llbracket \phi^2 \rrbracket + 3\langle \phi^2 \rangle^2$$



Therefore,

$$\begin{aligned} \phi_4 &= \llbracket \phi^4 \rrbracket = \phi^4 - 6G(0)\phi_2 + 3G(0)^2 \\ &= \phi^4 - 6G(0)\phi^2 - 3G(0)^2 \end{aligned}$$

## Scaling operators of Gaussian fixed-point

- To summarize, the scaling operators of Gaussian fixed point are

$$\phi_n \equiv [\phi^n]$$

- For the standard normalization, consider

$$\langle \phi_n(\mathbf{x}) \phi_n(\mathbf{y}) \rangle = \sum_{\text{all connection patterns}} \langle \phi(\mathbf{x}) \phi(\mathbf{y}) \rangle^n = N! G^n(r) = \frac{N!}{r^{2nx}}.$$

Therefore,  $\hat{\phi}_n \equiv \frac{1}{\sqrt{N!}} \phi_n$  is the normalized scaling operator.

- The scaling dimension is obviously

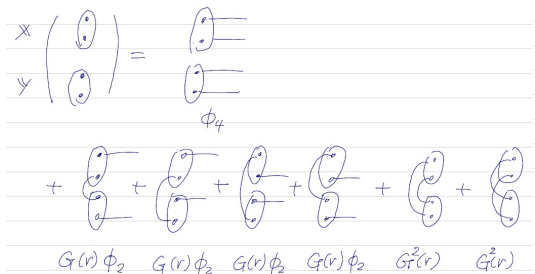
$$x_n \equiv nx = n(d-2)/2$$

- The scaling operators are orthogonal to each other

$$\langle \hat{\phi}_m(\mathbf{x}) \hat{\phi}_n(\mathbf{y}) \rangle = \frac{\delta_{mn}}{r^{2x_m}}$$

# Expansion of $\phi_2(\mathbf{x})\phi_2(\mathbf{y})$

- Consider product of two operators  $\phi_2(\mathbf{x})\phi_2(\mathbf{y})$ .

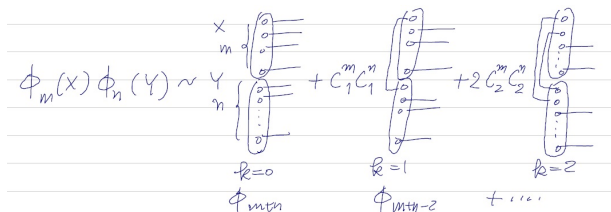


From this diagram, we obtain

$$\begin{aligned} \phi_2(\mathbf{x})\phi_2(\mathbf{y}) &\sim \phi_4\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) + 4\langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle\phi(\mathbf{x})\phi(\mathbf{y}) + 2\langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle^2 \\ &= \phi_4\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) + 4G(r)\phi_2\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) + 2G^2(r). \end{aligned}$$

# OPE of Gaussian fixed-point

- We can generalize the product  $\phi_2\phi_2$  to general two operators.



$$\begin{aligned}
 \phi_m(\mathbf{x})\phi_n(\mathbf{y}) &\sim \sum_{k=0}^{(m+n)/2} \binom{m}{k} \binom{n}{k} k! G^k(r) \phi_{m+n-2k} \left( \frac{\mathbf{x} + \mathbf{y}}{2} \right) \\
 &= \sum_l \frac{c_{mn}^l}{|\mathbf{x} - \mathbf{y}|^{x_m + x_n - x_l}} \phi_l \left( \frac{\mathbf{x} + \mathbf{y}}{2} \right) \\
 &\quad (c_{mn}^l = \binom{m}{k} \binom{n}{k} k! \quad (k \equiv \frac{m+n-l}{2}))
 \end{aligned}$$



## Summary

- Generally, we can expand the product of the two scaling operators in terms of scaling operators (OPE), which takes the form

$$\phi_m(\mathbf{x})\phi_n(\mathbf{y}) = \sum_l \frac{c_{mn}^l}{|\mathbf{x} - \mathbf{y}|^{x_m+x_n-x_l}} \phi_l\left(\frac{\mathbf{x} + \mathbf{y}}{2}\right).$$

- The constants  $c_{mn}^l$  are universal quantities (provided that the scaling operators are properly normalized).
- For the Gaussian model, the scaling operator can be explicitly defined by the normal-ordering as  $\phi_n \equiv \llbracket \phi^n \rrbracket$ , and its scaling dimension is  $x_n \equiv nx = n(d-2)/2$ .
- The OPE at the Gaussian fixed-point is characterized by

$$c_{mn}^l = \binom{m}{k} \binom{n}{k} k! \quad \left( k \equiv \frac{m+n-l}{2} \right)$$

## Exercise

- For the Gaussian model, obtain  $\phi_5$  in terms of  $\phi^k$  ( $k = 1, 2, 3, \dots$ ), following the same argument as in the lecture.