Lecture 7: Consequences of Renormalization Group

Naoki KAWASHIMA

ISSP, U. Tokyo

June 3, 2019

In this lecture, we see ...

- The free energy (and therefore all the quantities derived from it) can be expressed as the sum of a singular part and a regular part.
- The critical phenomena can be systematically derived from the singular part of the free energy.
- By RG flow diagram, we can understand cross-over phenomena, which is the scale-dependent critical phenomena.
- From RG, we can derive "finite-size scaling (FSS)," which is useful in estimating scaling dimensions through numerical simulation of finite systems.

[7-1] Singular part of free energy

- The free energy of a finite system can be split into two non-singular parts: the first part is purely extensive, whereas the second is RGT invariant and becomes singular in the thermodynamic limit. The latter is called the singular part of the free energy (though it is non-singular for finite systems).

Singular part of free energy

- As we see later, the RGT invariant function produces a singularity that explains critical behaviors.
- However, the free energy itself cannot be RGT invariant at the critical point due to cotribution from short-range fluctuations.
- These observations motivate the following form for the free energy:

$$F(\mathbf{K}, L) = F_{\mathrm{s}}(\mathbf{K}, L) + L^d \gamma(\mathbf{K}) \quad (1)$$

where F_{s} is RGT-invariant

$$F_{\mathrm{s}}(\acute{\mathbf{K}}, \acute{L}) = F_{\mathrm{s}}(\mathbf{K}, L) \quad (2)$$

and $\gamma(\mathbf{K})$ is a non-singular function of $\mathbf{K}$

Example: 1D Ising model (1)

- The partition function can be expressed with the transfer matrix as

$$Z = \mathrm{Tr}\, T^L \quad \left(T_{S_1,S_2} \equiv e^{KS_1S_2 + h(S_1+S_2)/2} = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix} \right)$$

- The eigenvalues of T are

$$\lambda_\pm \equiv e^K \left(\cosh h \pm \sqrt{\sinh^2 h + e^{-4K}} \right) \qquad (3)$$

- The correlation length is then

$$\xi^{-1} = -\log \frac{\lambda_-}{\lambda_+} \approx 2\sqrt{h^2 + t^2} \qquad \left(t \equiv e^{-2K} \right). \qquad (4)$$

- $Z = \mathrm{Tr}\, T^L = \lambda_+^L + \lambda_-^L = \lambda_+^L \left(1 + \left(\frac{\lambda_-}{\lambda_+} \right)^L \right)$

Example: 1D Ising model (2)

- $F = Lf + \Delta F \quad \left(f \equiv -\log \lambda_+,\ \Delta F \equiv -\log\left(1 + e^{-L/\xi}\right)\right)$
- Notice that ΔF is obviously RGT-invariant. However, we cannot take f and ΔF as γ and $F_{\rm s}$, respectively, because f is singular.
- Notice also that, for $\xi \gg L$, we have $\Delta F \approx -\log 2 + \dfrac{L}{2\xi}$.
- Therefore, by subtracting $L/2\xi$ from ΔF and add it to Lf, we can make both terms non-singular*, while keeping ΔF RGT-invariant:

$$F = F_{\rm s} + L\gamma \tag{5}$$

with $F_{\rm s} \equiv \Delta F - \dfrac{L}{2\xi} \ \left(f_{\rm s} \equiv \lim_{L\to\infty} \dfrac{F_{\rm s}}{L^d} = -\dfrac{L}{2\xi}\right)$, and $\gamma \equiv f + \dfrac{1}{2\xi}$

($*$) Since F is the free energy of a finite system, it must be regular. Therefore, regularity of γ automatically means regularity of $F_{\rm s}$ even if it produces a singularity in the $L \to \infty$ limit. In addition, strictly speaking, γ is singular, but this singularity is physically uninportant, because it can be removed by adding a constant to the Hamiltonian.

[7-2] Scaling form

- It is convenient to introduce the scaling form of the singular part of the free energy.
- From it, we can systematically derive various scaling relations.

Finite system

- The RGT invariance, $F_{\rm s}(\mathbf{K}, L) = F_{\rm s}(\acute{\mathbf{K}}, \acute{L})$, can be rewritten in terms of scaling field, u_μ,

$$F_{\rm s}(u_1, u_2, \cdots, L) = F_{\rm s}(u_1 b^{y_1}, u_2 b^{y_2}, \cdots, L/b). \tag{6}$$

- By setting $b = L/L_0$ where L_0 is some constant length scale, and dropping the L_0 dependence of the function, we may write

$$F_{\rm s}(u_1, u_2, \cdots, L) = \tilde{F}_{\rm s}(u_1 L^{y_1}, u_2 L^{y_2}, \cdots), \tag{7}$$

which is called the "scaling form" of $F_{\rm s}$.

Infinite system

- Another way of rewriting F_{s} is

$$
\begin{aligned}
F_{\mathrm{s}}(u_1, u_2, \cdots, L) &= L^d f_{\mathrm{s}}(u_1, u_2, \cdots, L) \\
&= (L/b)^d f_{\mathrm{s}}(u_1 b^{y_1}, u_2 b^{y_2}, \cdots, L/b).
\end{aligned}
$$

- Let us assign a special role to the first scaling field, u_1, which we assume to be relevant, and denote it as t ($t \equiv u_1$).
- By taking b so that $tb^{y_t} = t_0$ is a constant,

$$
f_{\mathrm{s}}(u_1, u_2, \cdots, L) = t^{\frac{d}{y_t}} f_{\mathrm{s}}(t_0, u_2 t^{-y_2/y_1}, \cdots, L t^{1/y_t})
$$

- In the thermodynamic limit, the L dependence on the both side should vanish. Then, by also droping the t_0 dependence,

$$
f_{\mathrm{s}}(u_1, u_2, u_3, \cdots) = t^{\frac{d}{y_t}} \tilde{f}_{\mathrm{s}} \left(u_2 t^{-\frac{y_2}{y_1}}, u_3 t^{-\frac{y_3}{y_1}}, \cdots \right) \tag{8}
$$

which is called the scaling form of the free energy density.

Singularity of various quantities (1)

- We can derive various scaling properties from (7) (or (8)).
- Below, we consider only the vicinity of the critical point which allows us to set all irrelevant fields zero.
- As an example, we consider the case where we have only two relevant fields, $t \equiv u_1$ and $h \equiv u_2$ (like $t \propto T - T_c$ and $h \propto H$ in the Ising model). So, our singular part of the free energy becomes

$$F_{\mathrm{s}}(t, h, L) = \tilde{F}_{\mathrm{s}}(tL^{y_t}, hL^{y_h}) \tag{9}$$

Singularity of various quantities (2)

$$F_{\mathrm{s}}(t,h,L) = \tilde{F}_{\mathrm{s}}(tL^{y_t}, hL^{y_h})$$

- For the "specific heat", we have

$$c(t,L) \propto -\frac{1}{L^d}\left(\frac{\partial^2 F_{\mathrm{s}}}{\partial t^2}\right)_{h\to 0} \sim -L^{-d+2y_t}\tilde{F}_{\mathrm{s}}^{(2,0)}(tL^{y_t},0)$$

$$\left(\tilde{F}_{\mathrm{s}}^{(m,n)} \equiv \frac{\partial^m}{\partial t^m}\frac{\partial^n}{\partial h^n}\tilde{F}_{\mathrm{s}}\right)$$

$$\sim L^{2y_t-d}\left(tL^{y_t}\right)^{-\frac{2y_t-d}{y_t}} \times \left(-\left(tL^{y_t}\right)^{\frac{2y_t-d}{y_t}}\tilde{F}_{\mathrm{s}}^{(2,0)}(tL^{y_t},0)\right)$$

$$= t^{-\frac{2y_t-d}{y_t}}\tilde{c}(tL^{y_t}) \qquad \left(\tilde{c}(x) \equiv -x^{\frac{2y_t-d}{y_t}}\tilde{F}_{\mathrm{s}}^{(2,0)}(x,0)\right)$$

- Since $\lim_{L\to\infty} c(t,L)$ is independent of L, $c(t,\infty) \propto t^{-\alpha}$ where

$$\alpha = \frac{2y_t-d}{y_t} = 2 - d\nu \quad \left(\nu \equiv \frac{1}{y_t}\right) \tag{10}$$

Singularity of various quantities (3)

- For "magnetization", we have

$$m \propto -\frac{1}{L^d}\left(\frac{\partial F_{\rm s}}{\partial h}\right)_{h\to 0} = -L^{-d+y_h} F_{\rm s}^{(0,1)}(tL^{y_t},0)$$

$$= t^{\frac{d-y_h}{y_t}}\tilde{m}(tL^{y_t}) \propto t^{\beta}$$

$$\text{with} \quad \beta \equiv \frac{d-y_h}{y_t} \tag{11}$$

- For "magnetic susceptibility", we have

$$\chi \propto -\frac{1}{L^d}\left(\frac{\partial^2 F_{\rm s}}{\partial h^2}\right)_{h\to 0} = -L^{-d+2y_h} F_{\rm s}^{(0,2)}(tL^{y_t},0)$$

$$= t^{-\frac{2y_h-d}{y_t}}\tilde{\chi}(tL^{y_t}) \propto t^{-\gamma}$$

$$\text{with} \quad \gamma \equiv \frac{2y_h-d}{y_t} \tag{12}$$

Scaling relations

- From (10), (11) and (12),

$$\alpha + 2\beta + \gamma = 2. \qquad \text{(Rushbrooke)} \tag{13}$$

- Similarly, we can also derive that

$$\gamma = \beta(\delta - 1) \qquad \text{(Griffiths)} \tag{14}$$

where δ is the exponent that characterize the magnetic-field dependence of the magnetization at the critical temperature,

$$m(t=0,h) \propto h^{1/\delta}$$

[7-3] Cross-over phenomena

- A cross-over phenomenon is the behavior of the system in which a weak but relevant scaling field manifests itself.
- We can understand it from the scaling form.
- We can also derive the form of the phase boundary near the critical point.

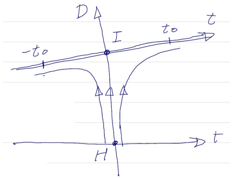
Example: Heisenberg model with anisotropy (1)

- 3D classical Heisenberg model

$$\mathcal{H} = -J \sum_{(ij)} \mathbf{S}_i \cdot \mathbf{S}_j - D \sum_{(ij)} S_i^z S_j^z$$

where $\mathbf{S} \equiv (S_i^x, S_i^y, S_i^z)^{\mathrm{T}}$ $(|\mathbf{S}| = 1)$.

- If $D = 0$, the system has a critical point, corresponding to the fixed point "H".
- The anisotropic operator is relevant at H.
- If the anisotoropy is strong enough, we can regard the system as an Ising model, whose critical point is represented by "I".
- Accordingly, there is a RG trajectory (critical line) starting from H and ending at I.



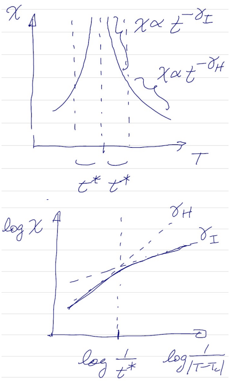
Example: Heisenberg model with anisotropy (2)

- By (8), the free energy around "H" is

$$f_s(t, D) = t^{\frac{d}{y_t}} \tilde{f}_s\left(Dt^{-\phi}\right) \qquad (15)$$

where ϕ is cross-over exponent $\phi \equiv \dfrac{y_D}{y_t}$.

- Eq.(15) can be re-written as $f_s = t^{\frac{d}{y_t}} \tilde{f}_s(t/t^*(D))$ with a "cross-over temperature" $t^*(D) \propto D^{1/\phi}$. Then, f_s behaves like an isotropic Heisenberg model $(D = 0)$ when $t \gg t^*$, whereas it qualitatively deviates from the Heisenberg-like behavior when $t \ll t^*$.

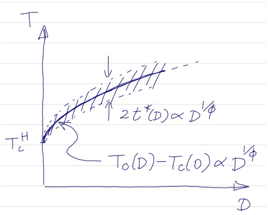


$$\gamma_{3\mathrm{DI}} = 1.237075(10)$$
$$\gamma_{3\mathrm{DH}} \approx 1.35(*)$$
$$\gamma_{3\mathrm{DXY}} = 1.3177(5)$$

(*) Kaupuzs, cond-mat/0101156

Example: Heisenberg model with anisotropy (3)

- Now, we consider the shape of the phase boundary in the $D-T$ phase diagram.
- We again use $f_{\mathrm{s}}(t,D) = t^{d/y_t^H}\tilde{f}_{\mathrm{s}}(Dt^{-\phi})$, where $\phi \equiv y_D^H/y_t^H$. where we put superscript "H" to make it clear that y_t^H is the value at H.
- Wnen $D>0$, the system should show the Ising-like critical behavior. Then, we obtain
$$f_{\mathrm{s}}(t,D) \sim (t - t_c(D))^{d/y_t^I}.$$



$\phi \approx 1.2$ for 3D Heisenberg model.

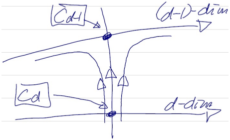
- Now, to satisfy both of these forms at the same time, f_{s} must have the following form near the criticality.

$$f_{\mathrm{s}} \propto t^{\frac{d}{y_t^H}}\left(tD^{-\frac{1}{\phi}} - x_0\right)^{\frac{d}{y_t^I}} \propto D^{\frac{d}{\phi}\left(\frac{1}{y_t^H}-\frac{1}{y_t^I}\right)}\left(t - x_0 D^{\frac{1}{\phi}}\right)^{\frac{d}{y_t^I}}$$

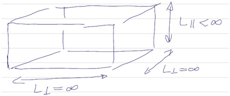
Therefore, $t_c(D) \propto D^{1/\phi} = D^{y_t^H/y_t^I}$

Dimensional crossover

- Some systems have phase transitions even when the size is finite in one direction. However, the critical properties are different from the case where the system is infinite in all directions.



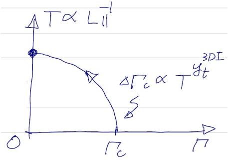
- Though the system size is not a "field" in the conventional terminology, we can treat $(\text{size})^{-1}$ as if it were a relevant field.



- In doing so, the scaling eigenvalue of $D \equiv L_{\|}^{-1}$, where $L_{\|}$ is the size that is finite, is obviously 1.

- Therefore, we have $\phi = 1/y_t$ for the crossover exponent, which leads to $t^* \propto L_{\|}^{-y_t}$ for the crossover temperature, and $t_c(L_{\|}) \propto L_{\|}^{-y_t}$ for the transition temperature near the $L_{\|} = \infty$ critical point.

Quantum critical point



- By Feynman's path integral formulation, d-dimensional quantum system can be represented as $(d+1)$-dimensional classical system with size $1/T$ in the new direction.
- In some special cases, the extra dimension, called the "imaginary time", is essentially equivalent to one of the spatial directions.
- For example, the 2-dimensional transverse field Ising model $\mathcal{H} = -J \sum_{(ij)} S_i^z S_i^z - \Gamma \sum_i S_i^x$ has a quantum phase transition at $T = 0$ and $\Gamma = \Gamma_c$, and it can be mapped to 3-dimensional classical Ising model with the size $1/T$ in the 3rd dimension.
- Then, we can apply the dimensional cross-over to this system:

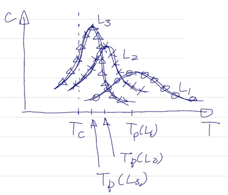
$$t^*(T) \propto t_c(T) \propto L_{\|}^{-y_t^{3DI}} \propto T^{y_t} \qquad (16)$$

where $t \equiv \Gamma - \Gamma_c$ and y_t is for the 3D Ising model.

[7-4] Finite-size scaling

- As we have seen, the RGT-invariant quantity can be used to characterize a critical phenomena.
- We'll see a practical way for obtaining the critical indices (scaling dimensions).
- We can define such a computable RGT-invariant quantity as a difference in the free energy of two system-sizes.

Specific heat (1)



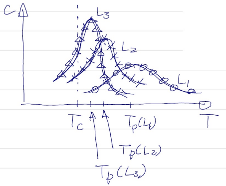
- Suppose we have obtained $F_{\rm s}(\mathbf{K}, L)$ as a function of $\mathbf{K}$ and L, and it has the form $F_{\rm s}(\mathbf{K}, L) = \tilde{F}_{\rm s}(tL^{y_t}, hL^{y_h}, \cdots)$.
- From $\tilde{F}_{\rm s}$, the scaling form of the specific heat is

$$c \approx \frac{-T}{L^d}\frac{\partial^2 F_{\rm s}}{\partial T^2} \sim L^{2y_t - d}\tilde{c}(tL^{y_t}). \qquad (17)$$

- If $c(T, L)$ diverges at the critical point for $L \to \infty$, we expect that c has a peak around $T \approx T_{\rm c}$ even if L is finite.
- This is compatible with (17) only if $\tilde{c}$ has a peak itself.

Specific heat (2)

$$c(T,L) \sim L^{2y_t-d}\tilde{c}(tL^{y_t}).$$



- Suppose $\tilde{c}(x)$ has a peak at $x = x_p$. It means that $c(T,L)$ has a peak when $tL^{y_t} = x_p$.
- Let $T_c(L)$ be the temperature at which $c(T,L)$ has the peak. Then,

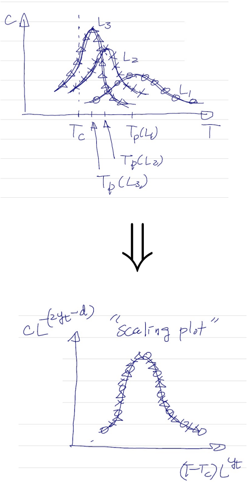
$$T_c(L) - T_c \propto t_c(L) \propto L^{-y_t}. \qquad (18)$$

- The height of the peak also carries some information on the critical behavior, i.e., it is proportional to

$$c(T_c(L),L) \propto L^{2y_t-d} \qquad (19)$$

Specific heat (3)

- More directly, by plotting c/L^{2y_t-d} against $(T-T_{\rm c})L^{y_t}$, we expect that curves corresponding to varying system sizes fall on top of each other.
- Of course, to do this, we need to choose the right values for $T_{\rm c}$ and y_t, which we do not know initially.
- We can fix these values by some trials-and-errors, like using an analog camera and adjusting the focus.



Remark: Practical substitute of F_{s}

- So far, we have been implicitly assuming that we can compute F_{s}.
- But it is not usually true even for finite systems. That's why people often simply use F itself in the place of F_{s} in numerical calculation.
  (This is equivalent to using the specific heat itself instead of its singular part.)
- This "approximation" is bad when the divergence is "weak."
- We had better use the following quantity, not F, in the place of F_{s}:

$$\begin{aligned}\Delta_b F(\mathbf{K}, L) &\equiv (b^d F(\mathbf{K}, L/b) - F(\mathbf{K}, L))/(b^d - 1) \\ &= (b^d F_{\mathrm{s}}(\mathbf{K}, L/b) - F_{\mathrm{s}}(\mathbf{K}, L))/(b^d - 1).\end{aligned}$$

  The last expression tells us that $\Delta_b F$ is RGT-invariant, i.e., free from the regular part, while the second expression is computable.

Exercise

- Consider the 1-dimensional q-state Potts model. Following the similar argument as in the lecture, obtain the singular part of the free energy.
- Consider $S = 1$ Ising model which is described by the same form of the Hamiltonian as the conventional Ising model, whereas each spin variable takes one of three values, $-1, 0, 1$ instead of two. Confirm that $\xi f_s = -1/2$ for this model, the same as the $S = 1/2$ Ising model.
- Derive Griffiths' scaling relation (14).