Lecture 3: ϕ^4 theory and Ornstein-Zernike form

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To begin with ...

- The mean-field theory discussed in the previous section does not tell us about the spatial correlation.
- In this lecture, starting from the Ising model, we derive ϕ^4 model, which, we expect, the same long-range behavior as the Ising model.
- We then apply the GBF variational approximation to the ϕ^4 Hamiltonian, to obtain the mean-field expression for the two-point correlation function. (Ornstein-Zernike form)

[3-1] ϕ^4 field theory

- We first see a very "hand-waving" derivation of ϕ^4 field theory starting from the Ising model and using the coarse-graining.
- We next see an alternative derivation which looks less hand-waving, based on the Hubbard-Stratonovich transformation.
- Since the ϕ^4 theory is obtained by the coarse-graining of the Ising model, they are supposed to share the same long-range behavior, while they may differ quantitatively for short-range physics.
- In particuar, we expect, ϕ^4 model belongs to the same universality class as the Ising model, as has been verified by a number of arguments and numerical calculations.

A hand-waving derivation by coarse-graining (1)

- Let us consider the Ising model on d-dimensional hyper-cubic lattice. (Hereafter, we use symbols like $\mathbf{r}$ and $\mathbf{R}$ to spacify lattice points in stead of i and j.)
- Divide the whole lattice into cells of size ab, where a is the lattice constant, and denote the one located at $\mathbf{R}$ as $\Omega_{ab}(\mathbf{R})$. $(b \gg 1)$
- Consider the cell average of spins

$$\phi_{\mathbf{R}} = \left(\frac{1}{b}\right)^d \sum_{\mathbf{r}\in\Omega_{ab}(\mathbf{R})} S_{\mathbf{r}} \tag{1}$$

- Consider the coarse-grained Hamiltonian $\tilde{\mathcal{H}}$ defined as

$$e^{-\tilde{\mathcal{H}}(\phi)} \equiv \sum_{\mathbf{S}} \Delta(\mathbf{S}|\phi) e^{-\mathcal{H}(\mathbf{S})}$$

where $\phi \equiv \{\phi_{\mathbf{R}}\}$, $\mathbf{S} \equiv \{S_{\mathbf{r}}\}$, and $\Delta(\mathbf{S}|\phi)\,(=0,1)$ takes 1 if and only if the condition (1) is satisfied for all cells.

A hand-waving derivation by coarse-graining (2)

- Let us guess, by intuition, what $\tilde{\mathcal{H}}$ should be like.
- There must be two parts: a single-cell part reflecting the physics inside each cell and a multiple-cell part for interactions.
- For one-cell part, the entropic effect gives rise to ϕ^2 and ϕ^4 terms (as in means-field approx.) whereas the interaction will produce $-\phi^2$.
- For multiple-cell part, the interaction among cells is represented by terms that depend on the gradient, $\nabla_{\mathbf{R}}\phi$. The mirror-image symmetry allows only even order terms. So, we expect $|\nabla\phi|^2$ in the lowest order.
- Putting these together and including the Zeeman term, we obtain

$$\tilde{H}(\phi) \equiv a^d \sum_{\mathbf{R}} \left(\rho|\nabla\phi|^2 + t\phi^2 + u\phi^4 - h\phi\right) \quad (2)$$

$$= \int_a^L d^d\mathbf{R} \left(\rho|\nabla\phi|^2 + t\phi^2 + u\phi^4 - h\phi\right) \quad (3)$$

(ρ, u are positive constants. t can be either positive or negative, depending on the temperature.)

Derivation by the Hubbard-Stratonovich transformation

$$Z_{\rm Ising} = \sum_S e^{K\sum_{(\mathbf{r},\mathbf{r}')} S_\mathbf{r} S_{\mathbf{r}'}} = \sum_S e^{\frac{K}{2}\mathbf{S}^\mathsf{T} C \mathbf{S} - \frac{cKN}{2}} \quad \left(C_{\mathbf{r},\mathbf{r}'} = \begin{cases} c & (|\mathbf{r}-\mathbf{r}'|=0) \\ 1 & (|\mathbf{r}-\mathbf{r}'|=a) \\ 0 & (\text{otherwise}) \end{cases} \right)$$

$$\overset{*}{=} \sum_S \int D\phi\, e^{-\frac{1}{2K}\phi^\mathsf{T} C^{-1}\phi + \phi\mathbf{S}} \quad \text{(HS transformation)}$$

$$= \int D\phi\, e^{-\frac{1}{2K}\phi^\mathsf{T} C^{-1}\phi} \prod_\mathbf{r} (2\cosh\phi_\mathbf{r})$$

$$\overset{*}{\approx} \int D\phi\, e^{-K^{-1}\sum_\mathbf{r}(\alpha\phi_\mathbf{r}^2 + \beta(\nabla\phi_\mathbf{r})^2)} e^{-\sum_\mathbf{r}(-\frac{1}{2}(\phi_\mathbf{r})^2 + \frac{1}{12}(\phi_\mathbf{r})^4)}$$

$$= \int D\phi\, e^{-\tilde{\mathcal{H}}(\phi)} = Z_{\phi^4}$$

$$\tilde{\mathcal{H}}(\phi) = \sum_\mathbf{r} \left(\frac{\beta}{K}|\nabla\phi_\mathbf{r}|^2 + \left(\frac{\alpha}{K} - \frac{1}{2}\right)|\phi_\mathbf{r}|^2 + \frac{1}{12}|\phi_\mathbf{r}|^4 \right)$$

Supplement: Hubbard-Stratonovich transformation (1)

For an arbitrary positive definite symmetric matrix A and a vector $\mathbf{B}$, we can show the following,

$$\begin{aligned}
&\int D\phi\, e^{-\frac{1}{2}\sum_{\mathbf{r},\mathbf{r}'} A_{\mathbf{r},\mathbf{r}'}\phi_{\mathbf{r}}\phi_{\mathbf{r}'} + \sum_{\mathbf{r}} B_{\mathbf{r}}\phi_{\mathbf{r}}} \\
&\quad = \int D\phi\, e^{-\frac{1}{2}\phi^{\mathsf{T}} A\phi + \mathbf{B}^{\mathsf{T}}\phi} \\
&\quad = \int D\boldsymbol{\xi} |A|^{-1/2} e^{-\frac{1}{2}\boldsymbol{\xi}^{\mathsf{T}}\boldsymbol{\xi} + \boldsymbol{\eta}^{\mathsf{T}}\boldsymbol{\xi}} \quad (\boldsymbol{\xi} \equiv A^{1/2}\phi,\ \boldsymbol{\eta} \equiv A^{-1/2}\mathbf{B}) \\
&\quad = \int D\boldsymbol{\xi} |A|^{-1/2} e^{-\frac{1}{2}(\boldsymbol{\xi}-\boldsymbol{\eta})^2 + \frac{1}{2}(\boldsymbol{\eta})^2} \\
&\quad = (2\pi)^{\frac{N}{2}} |A|^{-1/2} e^{\frac{1}{2}(\boldsymbol{\eta})^2} = (2\pi)^{\frac{N}{2}} |A|^{-1/2} e^{\frac{1}{2}\mathbf{B}^{\mathsf{T}} A^{-1}\mathbf{B}}
\end{aligned}$$

By taking KC for A^{-1} and $\mathbf{S}$ for $\mathbf{B}$,

$$e^{\frac{K}{2}\mathbf{S}^{\mathsf{T}} C\mathbf{S}} \sim \int D\phi\, e^{-\frac{1}{2K}\phi^{\mathsf{T}} C^{-1}\phi + \phi^{\mathsf{T}}\mathbf{S}}$$

Supplement: Hubbard-Stratonovich transformation (2)

The matrix C is defined as $C \equiv cI + \Delta$ where Δ is the connection matrix

$$\Delta_{\mathbf{r},\mathbf{r}'} \equiv \begin{cases} 1 & (|\mathbf{r}-\mathbf{r}'| = a) \\ 0 & (\text{Otherwise}) \end{cases}$$

We need its inverse, which we can compute as

$$\begin{aligned} C^{-1} &= \frac{1}{c}\left(I + \frac{1}{c}\Delta\right)^{-1} \\ &= \frac{1}{c}\left(I - \frac{1}{c}\Delta + \frac{1}{c^2}\Delta^2 + \cdots\right) \end{aligned}$$

This decays exponentially as a function of distance; truncation would not change things qualitatively, leading to what have been used in the main text

$$C^{-1} \approx \frac{1}{c}I - \frac{1}{c^2}\Delta \quad \left(\phi^{\mathsf{T}} C^{-1}\phi \approx \sum_{\mathbf{r}}\left(\frac{1}{c}(\phi_{\mathbf{r}})^2 + \frac{1}{2c^2}(\nabla\phi)^2\right)\right)$$

The meaning of ϕ in the HS derivation

For an arbitrary vector $\boldsymbol{\xi}$, we have

$$\langle \boldsymbol{\xi}^{\mathsf{T}} \mathbf{S} \rangle_{\text{Ising}} = Z_0^{-1} \frac{\partial}{\partial h} \sum_{\mathbf{S}} e^{\mathbf{S}^{\mathsf{T}} K \mathbf{S} + h \boldsymbol{\xi}^{\mathsf{T}} \mathbf{S}}$$

$$= Z_0^{-1} \frac{\partial}{\partial h} \int D\phi \sum_{\mathbf{S}} e^{-\frac{1}{2}\phi^{\mathsf{T}} K^{-1} \phi + \phi^{\mathsf{T}} \mathbf{S} + h \boldsymbol{\xi}^{\mathsf{T}} \mathbf{S}} \quad \text{(HS transformation)}$$

$$= Z_0^{-1} \frac{\partial}{\partial h} \int D\phi' \sum_{\mathbf{S}} e^{-\frac{1}{2}(\phi' - h\boldsymbol{\xi})^{\mathsf{T}} K^{-1} (\phi' - h\boldsymbol{\xi}) + \phi'^{\mathsf{T}} \mathbf{S}} \quad (\phi' \equiv \phi + h\boldsymbol{\xi})$$

$$= Z_0^{-1} \frac{\partial}{\partial h} \int D\phi' \sum_{\mathbf{S}} e^{-\frac{1}{2}\phi'^{\mathsf{T}} K^{-1} \phi' + h \boldsymbol{\xi}^{\mathsf{T}} K^{-1} \phi' + \phi'^{\mathsf{T}} \mathbf{S}} \quad \text{(Expand in } h\text{)}$$

$$= Z_0^{-1} \int D\phi \, e^{-\tilde{\mathcal{H}}_{\phi^4}(\phi)} \, \boldsymbol{\xi}^{\mathsf{T}} K^{-1} \phi = \langle \boldsymbol{\xi}^{\mathsf{T}} K^{-1} \phi \rangle_{\phi^4} \quad \text{(Remove } ' \text{)}$$

This means $\quad \phi_r \leftrightarrow \sum_{r'} K_{r,r'} S_r. \quad$ (A local sum of spins)

[3-2] Variational approximation to ϕ^4 model

- Similar to the Ising model, generally it is impossible to obtain the exact solution of ϕ^4 model by analytical means. So, we need some approximation. The simplest one is the mean-field type approximation as always.

- We will first move to the momentum space.

- Then, we will apply the GBF variational principle by taking the Gaussian theory as the trial Hamiltonian.

- As a result, we will obtain the mean-field evaluation of the spatial correlation function, which is called Ornstein-Zernike form.

Switching to the momentum space

Starting from (4), $\mathcal{H} = a^d \sum_{\mathbf{r}} \left(\rho |\nabla \phi_{\mathbf{r}}|^2 + t\phi_{\mathbf{r}}^2 + u\phi_{\mathbf{r}}^4 - h\phi_{\mathbf{r}} \right),$

by Fourier transformation $\phi_{\mathbf{r}} = L^{-d} \sum_{\mathbf{k}} e^{i\mathbf{kr}} \tilde{\phi}_{\mathbf{k}}$, we obtain

$$\begin{aligned} \mathcal{H} &= \frac{1}{L^d} \sum_{\mathbf{k}} (\rho k^2 + t) |\tilde{\phi}_{\mathbf{k}}|^2 \\ &+ \frac{u}{L^{3d}} \sum_{\mathbf{k}_1 \cdots \mathbf{k}_4} \delta_{\sum_{\mu=1}^4 \mathbf{k}_\mu, \mathbf{0}} \, \tilde{\phi}_{\mathbf{k}_1} \tilde{\phi}_{\mathbf{k}_2} \tilde{\phi}_{\mathbf{k}_3} \tilde{\phi}_{\mathbf{k}_4} - h\tilde{\phi}_{\mathbf{0}}. \end{aligned} \tag{4}$$

Switching to continuous wave numbers,

$$\begin{aligned} \mathcal{H} &= \int \frac{d^d \mathbf{k}}{(2\pi)^d} \, (\rho k^2 + t) \tilde{\phi}^*_{\mathbf{k}} \tilde{\phi}_{\mathbf{k}} \\ &+ u \int \frac{d^d \mathbf{k}_1 \cdots d^d \mathbf{k}_4}{(2\pi)^{4d}} \, \delta \left(\sum_{\mu} \mathbf{k}_\mu \right) \, \tilde{\phi}_{\mathbf{k}_1} \cdots \tilde{\phi}_{\mathbf{k}_4} - h\tilde{\phi}_{\mathbf{0}} \end{aligned} \tag{5}$$

Supplement: Convention (Fourier transformation)

In this lecture, we use the following conventions:

$$a = (\text{lattice constant}), \quad L = (\text{system size}), \quad N \equiv \frac{L^d}{a^d} = (\#\text{ of sites})$$

$$\tilde{\phi}_{\mathbf{k}} = \int_0^L d^d\mathbf{r}\, e^{-i\mathbf{k}\mathbf{r}} \phi_{\mathbf{r}} = a^d \sum_{\mathbf{r}} e^{-i\mathbf{k}\mathbf{r}} \phi_{\mathbf{r}}$$

$$\phi_{\mathbf{r}} = \int_{-\pi/a}^{\pi/a} \frac{d^d\mathbf{k}}{(2\pi)^d}\, e^{i\mathbf{k}\mathbf{r}} \tilde{\phi}_{\mathbf{k}} = L^{-d} \sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}} \tilde{\phi}_{\mathbf{k}}$$

The tilde ˜ is often dropped when there is no fear of confusion.

$$G(\mathbf{r}', \mathbf{r}) \equiv \langle \phi_{\mathbf{r}'} \phi_{\mathbf{r}} \rangle, \quad G_{\mathbf{k}',\mathbf{k}} \equiv L^{-d} \langle \phi_{\mathbf{k}'} \phi_{\mathbf{k}} \rangle$$

For translationally and rotationally symmetric case,

$$G(\mathbf{r}', \mathbf{r}) = G(|\mathbf{r}' - \mathbf{r}|), \quad G_{\mathbf{k}',\mathbf{k}} = \delta_{\mathbf{k}'+\mathbf{k},\mathbf{0}} G_{|\mathbf{k}|}, \quad G_{|\mathbf{k}|} \equiv L^{-d} \langle |\phi_{\mathbf{k}}|^2 \rangle$$

GBF variational approximation (1)

Let us consider a trial Hamiltonian with variational parameter $\epsilon_{\mathbf{k}}$,

$$\mathcal{H}_0 \equiv \frac{1}{L^d} \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} |\phi_{\mathbf{k}}|^2 \tag{6}$$

$$Z_0 = \int D\phi \, e^{-\frac{1}{L^d} \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} |\phi_{\mathbf{k}}|^2} = \prod_{\mathbf{k}} \zeta_{\mathbf{k}}$$

$$\langle |\phi_{\mathbf{k}}|^2 \rangle_0 = \frac{L^d}{2\epsilon_{\mathbf{k}}}, \quad \zeta_{\mathbf{k}} \equiv \left(\frac{\pi L^d}{\epsilon_{\mathbf{k}}} \right)^{1/2}$$

$$E_0 = \frac{1}{L^d} \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \langle |\phi_{\mathbf{k}}|^2 \rangle_0 = \sum_{\mathbf{k}} \frac{\epsilon_{\mathbf{k}}}{L^d} \frac{L^d}{2\epsilon_{\mathbf{k}}} = \sum_{\mathbf{k}} \frac{1}{2} \quad \text{(Equipartition)}$$

$$-TS_0 = F_0 - E_0 = -\sum_{\mathbf{k}} \frac{1}{2} \log \frac{\pi L^d}{\epsilon_{\mathbf{k}}} = \frac{1}{2} \sum_{\mathbf{k}} \log \epsilon_{\mathbf{k}}$$

(Additive constants have been omitted.)

GBF variational approximation (2)

$$\langle \mathcal{H} \rangle_0 = \frac{1}{L^d} \sum_{\mathbf{k}} (\rho k^2 + t) \langle |\phi_{\mathbf{k}}|^2 \rangle_0 + \frac{u}{L^{3d}} \sum_{\mathbf{k}_1 \cdots \mathbf{k}_4} \delta_{\sum \mathbf{k}, \mathbf{0}} \langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \phi_{\mathbf{k}_4} \rangle_0$$

$$= \frac{1}{L^d} \sum_{\mathbf{k}} (\rho k^2 + t) \langle |\phi_{\mathbf{k}}|^2 \rangle_0 + \frac{3u}{L^{3d}} \sum_{\mathbf{k}, \mathbf{k}'} \langle |\phi_{\mathbf{k}}|^2 \rangle_0 \langle |\phi_{\mathbf{k}'}|^2 \rangle_0 \quad \text{(Wick)}$$

In terms of $G_{\mathbf{k}} \equiv L^{-d} \langle |\phi_{\mathbf{k}}|^2 \rangle_0 = (2\epsilon_{\mathbf{k}})^{-1}$, we obtain

$$F_{\mathrm{v}} = \langle \mathcal{H} \rangle_0 - TS_0 = \sum_{\mathbf{k}} (\rho k^2 + t) G_{\mathbf{k}} + \frac{3u}{L^d} \left(\sum_{\mathbf{k}} G_{\mathbf{k}} \right)^2 + \frac{1}{2} \sum_{\mathbf{k}} \log \epsilon_{\mathbf{k}}$$

Thus we have, $$f_{\mathrm{v}} = B + 3uA^2 + \frac{1}{2L^d} \sum_{\mathbf{k}} \log \epsilon_{\mathbf{k}}, \quad (7)$$

where $A \equiv \frac{1}{L^d} \sum_{\mathbf{k}} G_{\mathbf{k}}$, and $B \equiv \frac{1}{L^d} \sum_{\mathbf{k}} (\rho k^2 + t) G_{\mathbf{k}}$.

Stationary condition

$$0 = \frac{\partial F_v}{\partial \epsilon_{\mathbf{k}}} = \left(\rho k^2 + t + \sigma\right) \frac{\partial G_{\mathbf{k}}}{\partial \epsilon_{\mathbf{k}}} + \frac{1}{2\epsilon_{\mathbf{k}}}$$

$$\left(\sigma \equiv 6uA = \frac{6u}{L^d} \sum_{\mathbf{k}} G_{\mathbf{k}}\right)$$

$\cdots$ Spatial fluctuation shifts the transition point.

$$= \left(\rho k^2 + t + \sigma\right) \left(-\frac{1}{2\epsilon_{\mathbf{k}}^2}\right) + \frac{1}{2\epsilon_{\mathbf{k}}}$$

$$\Rightarrow \quad \epsilon_{\mathbf{k}} = \rho k^2 + t + \sigma = \rho(k^2 + \kappa^2) \quad \left(\kappa \equiv \sqrt{\frac{t+\sigma}{\rho}}\right)$$

Ornstein-Zernike form

$$G_k \propto \frac{1}{k^2 + \kappa^2}, \quad \kappa \propto \sqrt{T - T_c}$$

Supplement: Wick's theorem

Theorem 1 (Wick)

When the distribution function is gaussian, any multi-point correlator factorizes in pairs.

Example 2 (4-point correlator)

Ex: When the Hamiltonian is $\mathcal{H} = \frac{1}{2}\phi^{\mathsf{T}} A \phi$ with A being a positive definit matrix,

$$
\begin{aligned}
\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle &= \langle \phi_1 \phi_2 \rangle \langle \phi_3 \phi_4 \rangle + \langle \phi_1 \phi_3 \rangle \langle \phi_2 \phi_4 \rangle + \langle \phi_1 \phi_4 \rangle \langle \phi_2 \phi_3 \rangle \\
&= \Gamma_{12}\Gamma_{34} + \Gamma_{13}\Gamma_{24} + \Gamma_{14}\Gamma_{23}
\end{aligned}
$$

where $\Gamma \equiv A^{-1}$ and $\langle \cdots \rangle \equiv \dfrac{\int D\phi \, e^{-\mathcal{H}(\phi)} \ldots}{\int D\phi \, e^{-\mathcal{H}(\phi)}}$

Supplement: Proof of Wick's theorem

If we define $\Xi \equiv \int D\phi \, e^{-\frac{1}{2}\phi^{\mathsf{T}}A\phi + \boldsymbol{\xi}^{\mathsf{T}}\phi}$, the correlation function can be expressed as its derivatives,

$$\left\langle \phi_{k_1}\phi_{k_2}\cdots\phi_{k_{2p}} \right\rangle = \Xi^{-1} \left(\frac{\partial}{\partial \xi_{k_1}} \cdots \frac{\partial}{\partial \xi_{k_{2p}}} \Xi \right) \Bigg|_{\boldsymbol{\xi}\to\mathbf{0}} .$$

Now notice that $\Xi \propto e^{\frac{1}{2}\boldsymbol{\xi}^{\mathsf{T}}\Gamma\boldsymbol{\xi}}$, which can be expanded as

$$\Xi = 1 + \sum_{ij} \frac{\Gamma_{ij}}{2}\xi_i\xi_j + \frac{1}{2}\sum_{ij}\sum_{kl} \frac{\Gamma_{ij}}{2}\frac{\Gamma_{kl}}{2}\xi_i\xi_j\xi_k\xi_l + \cdots$$

Therefore, the $2p$-body correlation becomes

$$\frac{1}{p!}\sum_{i_1 j_1}\sum_{i_2 j_2}\cdots\sum_{i_p j_p} \frac{\Gamma_{i_1 j_1}}{2}\frac{\Gamma_{i_2 j_2}}{2}\cdots\frac{\Gamma_{i_p j_p}}{2}\delta_{\{k_1,k_2,\cdots,k_{2p}\},\{i_1,j_1,i_2,j_2,\cdots,i_p,j_p\}}$$

$$= \sum \Gamma_{i_1 j_1}\Gamma_{i_2 j_2}\ldots\Gamma_{i_p j_p} \quad (\text{Summation over all pairings of } \{k_1,\cdots,k_{2p}\}\ )$$

Real-space correlation function

$$G_{\mathbf{k}} = L^{-d}\langle|\phi_{\mathbf{k}}|^2\rangle = \frac{1}{2\epsilon_{\mathbf{k}}} = \frac{1}{2(\rho k^2 + t + \sigma)}$$

$$G(\mathbf{r}' - \mathbf{r}) \equiv \langle \phi_{\mathbf{r}'}\phi_{\mathbf{r}}\rangle = L^{-2d}\sum_{\mathbf{k},\mathbf{k}'} e^{i\mathbf{k}'\mathbf{r}'} e^{i\mathbf{k}\mathbf{r}} \langle \phi_{\mathbf{k}'}\phi_{\mathbf{k}}\rangle$$

$$= L^{-2d}\sum_{\mathbf{k},\mathbf{k}'} e^{i\mathbf{k}'\mathbf{r}'} e^{i\mathbf{k}\mathbf{r}} \delta_{\mathbf{k}'+\mathbf{k},\mathbf{0}} G_{\mathbf{k}} = L^{-2d}\sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{r}'-\mathbf{r})}\frac{L^d}{2\epsilon_{\mathbf{k}}}$$

$$G(\mathbf{r}) = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{e^{i\mathbf{k}\mathbf{r}}}{2\epsilon_{\mathbf{k}}} = \frac{1}{2}\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{e^{i\mathbf{k}\mathbf{r}}}{\rho k^2 + t + \sigma}$$

$$\overset{*}{\sim} \begin{cases} \frac{1}{\rho}\frac{\kappa^{d-2}}{(\kappa r)^{\frac{d-1}{2}}} e^{-\kappa r} \quad \left(\kappa r \gg 1,\ \kappa \equiv \sqrt{\frac{t+\sigma}{\rho}}\right) & (T > T_c) \\ \frac{1}{\rho}\frac{1}{r^{d-2}} & (T = T_c) \end{cases}$$

($* \cdots$ see supplement)

Mean-field values of ν and η

$$G(r) \sim \begin{cases} \frac{1}{\rho} \frac{\kappa^{d-2}}{(\kappa r)^{\frac{d-1}{2}}} e^{-\kappa r} & \left(\kappa r \gg 1,\ \kappa \equiv \sqrt{\frac{t+\sigma}{\rho}}\right) \quad (T > T_c) \\ \frac{1}{\rho} \frac{1}{r^{d-2}} & \quad (T = T_c) \end{cases}$$

Mean-field value of ν

For $T > T_c$, $\quad G(r) \propto \frac{1}{r^{\frac{d-1}{2}}} e^{-r/\xi}, \quad \xi \propto \frac{1}{|T - T_c|^{\nu}}, \quad \nu_{\mathrm{MF}} = \frac{1}{2}$

Mean-field value of η

At $T = T_c$, $\quad G(r) \propto \frac{1}{r^{d-2+\eta}}, \quad \eta_{\mathrm{MF}} = 0$

Supplement: Evaluation of the asymptotic form ($T > T_c$)

$$\int d\mathbf{k}\, \frac{e^{i\mathbf{k}\mathbf{r}}}{k^2+\kappa^2} = \int d\mathbf{k}\, e^{i\mathbf{k}\mathbf{r}} \int_0^\infty dt e^{-t(k+\kappa^2)}$$

$$= \int_0^\infty dt\, e^{-t\kappa^2} \int d\mathbf{k} e^{-tk^2+i\mathbf{r}\mathbf{k}}$$

$$= \int_0^\infty dt\, e^{-t\kappa^2} \int d\mathbf{k} e^{-t(\mathbf{k}-\frac{i}{2t}\mathbf{r})^2-\frac{\mathbf{r}^2}{4t}} = \int_0^\infty dt\, \left(\frac{\pi}{t}\right)^{\frac{d}{2}} e^{-\kappa^2 t-\frac{r^2}{4t}}$$

(Here we define u so that $t \equiv \frac{r}{2\kappa}u$ and $\kappa^2 t + \frac{r^2}{4t} = \frac{\kappa r}{2}(u+u^{-1})$.)

$$= \int_0^\infty du\, \left(\frac{\pi}{u}\right)^{\frac{d}{2}} \left(\frac{2\kappa}{r}\right)^{\frac{d}{2}-1} e^{-\frac{\kappa r}{2}(u+u^{-1})}$$

(For $\kappa r \gg 1$, we use $u+u^{-1} \approx 2+\epsilon^2$ where $\epsilon \equiv u-1$.)

$$\approx \pi^{\frac{d}{2}} \left(\frac{2\kappa}{r}\right)^{\frac{d}{2}-1} e^{-\kappa r} \left(\frac{2\pi}{\kappa r}\right)^{\frac{1}{2}} \qquad \sim \frac{\kappa^{d-2}}{(\kappa r)^{\frac{d-1}{2}}} e^{-\kappa r}$$

Supplement: Evaluation of the asymptotic form ($T = T_c$)

As before, we have

$$\int d\mathbf{k}\, \frac{e^{i\mathbf{kr}}}{k^2+\kappa^2} = \int_0^\infty dt\, \left(\frac{\pi}{t}\right)^{\frac{d}{2}} e^{-\kappa^2 t - \frac{r^2}{4t}}$$

Here, by setting $\kappa = 0$ ($T = T_c$),

$$= \int_0^\infty dt\, \left(\frac{\pi}{t}\right)^{\frac{d}{2}} e^{-\frac{r^2}{4t}}$$

(By defining $\eta \equiv \frac{r^2}{4t}$)

$$= \left(\frac{r^2}{4}\right)^{1-\frac{d}{2}} \pi^{\frac{d}{2}} \Gamma\left(\frac{d}{2}-1\right) \sim \frac{1}{r^{d-2}}$$

Gaussian MF approximation below T_c (1)

- To deal with the spontaneous magnetization below T_c, we must introduce a symmetry-breaking field η as a new variational parameter,

$$\mathcal{H}_0 = L^{-d} \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} |\phi_{\mathbf{k}}|^2 - \eta \phi_{\mathbf{k}=\mathbf{0}}$$

- It is, then, a little tedious but not hard to see that (7) is replaced by

$$f_{\mathrm{v}} \stackrel{*}{=} B + tm^2 + u(3A^2 + 6Am^2 + m^4) + \frac{1}{2L^d} \sum_{\mathbf{k}} \log \epsilon_{\mathbf{k}}, \qquad (8)$$

where $m \equiv \langle \phi_{\mathbf{r}} \rangle_0$ and, as before,

$$A \equiv \frac{1}{L^d} \sum_{\mathbf{k}} \frac{1}{2\epsilon_{\mathbf{k}}}, \quad B \equiv \frac{1}{L^d} \sum_{\mathbf{k}} \frac{\rho \mathbf{k}^2 + t}{2\epsilon_{\mathbf{k}}}$$

Gaussian MF approximation below T_c (2)

- From $\partial f_{\rm v}/\partial m = 0$, we obtain

$$
\begin{aligned}
& t + 6uA + 2um^2 = 0 \\
\text{or} \quad & m^2 = -\frac{t+\sigma}{2u} \quad (\sigma \equiv 6uA)
\end{aligned}
\tag{9}
$$

- From $\partial f_{\rm v}/\partial \epsilon_{\mathbf{k}} = 0$ ($\mathbf{k} \neq \mathbf{0}$), we obtain

$$\epsilon_{\mathbf{k}} = \rho k^2 + t + 6u(A + m^2).$$

Using (9), $\epsilon_{\mathbf{k}} = \rho k^2 - 2(t+\sigma) = \rho(k^2 + \acute{\kappa}^2) \quad \left(\acute{\kappa}^2 \equiv \frac{2(t+\sigma)}{\rho}\right)$

- Thus, we have obtained the Ornstein-Zernike type Green's function

$$G_k = \frac{1}{2\epsilon_{\mathbf{k}}} = \frac{1}{2(k^2 + \acute{\kappa}^2)} \quad (T < T_c)$$

The correlation length is $1/\sqrt{2}$ times smaller than the high-T side.

Supplement: Wick's theorem with symmetry-breaking field

For deriving (8), since the external field distorts the Gaussian distribution, which is the precondition to the Wick's theorem, we must apply the theorem to the fluctuation $\delta\phi_{\mathbf{r}} \equiv \phi_{\mathbf{r}} - \langle\phi_{\mathbf{r}}\rangle_0$, not ϕ itself. In the momentum space, by defining $\delta\phi_{\mathbf{k}} \equiv \phi_{\mathbf{k}} - \bar{\phi}_{\mathbf{0}}\delta_{\mathbf{k}}$ ($\delta_{\mathbf{k}} \equiv \delta_{\mathbf{k},\mathbf{0}}$, $\bar{\phi}_{\mathbf{0}} = L^d m$),

$$
\begin{aligned}
&\langle \phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\phi_{\mathbf{k}_3}\phi_{\mathbf{k}_4}\rangle_0 \\
&\quad = \langle(\bar{\phi}_0\delta_{\mathbf{k}_1} + \delta\phi_{\mathbf{k}_1})(\bar{\phi}_0\delta_{\mathbf{k}_2} + \delta\phi_{\mathbf{k}_2})(\bar{\phi}_0\delta_{\mathbf{k}_3} + \delta\phi_{\mathbf{k}_3})(\bar{\phi}_0\delta_{\mathbf{k}_4} + \delta\phi_{\mathbf{k}_4})\rangle_0 \\
&\quad = \bar{\phi}_0^4\delta_{\mathbf{k}_1}\delta_{\mathbf{k}_2}\delta_{\mathbf{k}_3}\delta_{\mathbf{k}_4} + \bar{\phi}_0^2\left(\delta_{\mathbf{k}_1}\delta_{\mathbf{k}_2}\langle\delta\phi_{\mathbf{k}_3}\delta\phi_{\mathbf{k}_4}\rangle_0 + 5 \text{ similar terms}\right) \\
&\quad + \left(\langle\phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\rangle_0\langle\phi_{\mathbf{k}_3}\phi_{\mathbf{k}_4}\rangle_0 + 2 \text{ similar terms}\right)
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} \delta_{\sum\mathbf{k}}\langle \phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\phi_{\mathbf{k}_3}\phi_{\mathbf{k}_4}\rangle_0 \\
&\quad = \bar{\phi}_0^4 + 6\bar{\phi}_0^2\sum_{\mathbf{k}_1}\langle\delta\phi_{\mathbf{k}_1}\delta\phi_{-\mathbf{k}_1}\rangle_0 + 3\sum_{\mathbf{k}_1,\mathbf{k}_3}\langle\phi_{\mathbf{k}_1}\phi_{-\mathbf{k}_1}\rangle_0\langle\phi_{\mathbf{k}_3}\phi_{-\mathbf{k}_3}\rangle_0
\end{aligned}
$$

Exercise

- Consider an Ising model with only 4 spins.

$$\mathcal{H} = -K(S_1S_2 + S_3S_4) - K'(S_1S_3 + S_2S_4 + S_1S_4 + S_2S_3)$$

By coarse-graining

$$\phi_1 \equiv \frac{1}{2}(S_1 + S_2) \text{ and } \phi_2 \equiv \frac{1}{2}(S_3 + S_4),$$

obtain the **exact** effective Hamiltonian in terms of ϕ_1 and ϕ_2, and verify the existence of terms proportional to ϕ^2, ϕ^4 and $|\nabla\phi|^2 (= (\phi_1 - \phi_2)^2)$, respectively. (If necessary, solve numerically by setting some numerical values of your choice to K and K'.)