Lecture 4: Introduction to Renormalization Group

Naoki KAWASHIMA

ISSP, U. Tokyo

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To begin with ...

- There are cases where we can rely on the mean-field theory even for the critical behavior. (Ginzburg criterion)
- In one dimension, we may be able to carry out coarse-graining and obtain correct critical behavior. However, in higher dimensions, similar approaches would not yield computationally tractable solutions.
- Real-space renormalization group transformation can be approximately carried out (Migdal-Kadanoff RG) and produces a non-trivial (non-MF) evaluates of critical exponents. However, they do not generally agree with the correct values, nor they provide a way to systematically improve the approximation.

[4-1] When can MF be valid? — Ginzburg criterion

- First, we will elucidate the meaning of the asymptotic validity and draw a general criterion.
- Then, we will check whether the mean-field theory satisfies the criterion in a self-consistent way.
- We will find that it is indeed self-consistent in some cases, but not in general. (Ginzburg criterion)

Asymptotic validity of MF approximation

- Consider a system just below the critical temperature, where there is a finite but small spontaneous magnetization.
- The mean-field (MF) description should be valid when the relative fluctuation is negligible, i.e., $\delta\phi_{\mathbf{r}} \ll \langle \phi_{\mathbf{r}} \rangle$
- Typically, this condition is **not** satisfied at the scale of lattice constant, e.g., for the Ising model, $\langle \phi_{\mathbf{r}} \rangle \approx 0$ and $\delta\phi_{\mathbf{r}} = \sqrt{\langle \delta\phi_{\mathbf{r}}^2 \rangle} \approx 1$.
- However, the MF description can still be qualitatively correct at larger length-scales relevant to the critical behavior, i.e., ξ.
- So, we consider the local average of ϕ, i.e., $\bar{\phi}_{\mathbf{R}} \equiv \frac{1}{b^d} \sum_{\mathbf{r} \in \Omega_b(\mathbf{R})} \phi_{\mathbf{r}}$
- The condition for asymptotic validity of MF is $\delta\bar{\phi}_{\mathbf{R}} \ll \langle \bar{\phi}_{\mathbf{R}} \rangle$ for some $b \sim \xi$.

Self-consistency of mean-field approximation (1)

- For $\langle\bar{\phi}\rangle$, below T_c, we have $\langle\bar{\phi}\rangle^2_{\rm MF} \sim m^2 \sim \dfrac{|\Delta t|}{u} \sim \dfrac{\rho}{u\xi^2}$
- For the amplitude of the fluctuation, we have

$$\langle(\delta\bar{\phi})^2\rangle_{\rm MF} = \left(\frac{a}{b}\right)^{2d} \sum_{\mathbf{r},\mathbf{r}'\in\Omega_b(\mathbf{R})} \langle\delta\phi_{\mathbf{r}'}\delta\phi_{\mathbf{r}}\rangle \overset{*}{\sim} \frac{\xi^2}{\rho b^d} \quad (* \text{ see supplement})$$

- Thus, the validity condition becomes $\dfrac{\rho}{u\xi^2} \gg \dfrac{1}{\rho\xi^{d-2}} \left(\xi^{d-4} \gg \dfrac{u}{\rho^2}\right)$
- For $d > 4$, the condition is asymptotically satisfied as one approaches the critical point, whereas it is not for $d < 4$.

Ginzburg criterion (Upper critical dimension)

The MF approximation to ϕ^4 model cannot be correct asymptotically below 4 dimensions while it can be correct above 4.

Supplement: MF estimate of fluctuation (1)

In Lecture 3, we saw

$$\langle \delta\phi_{\mathbf{r}'}\delta\phi_{\mathbf{r}}\rangle \sim \frac{1}{\rho}\frac{\kappa'^{d-2}}{(\kappa' r)^{\frac{d-1}{2}}} e^{-\kappa'|\mathbf{r}'-\mathbf{r}|} \qquad (\kappa' \approx \sqrt{-\Delta t})$$

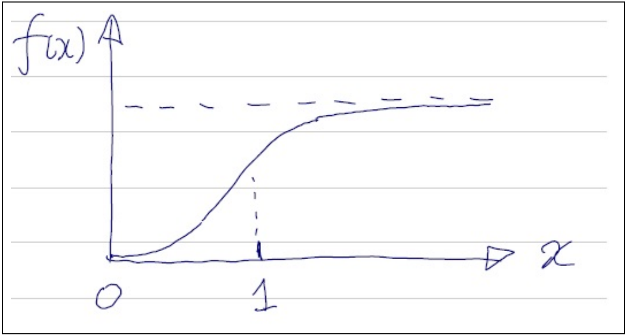
from which we obtain

$$\langle(\delta\bar{\phi})^2\rangle = \left(\frac{a}{b}\right)^{2d} \sum_{\mathbf{r},\mathbf{r}'\in\Omega_b(\mathbf{R})} \langle \delta\phi_{\mathbf{r}'}\delta\phi_{\mathbf{r}}\rangle \sim \left(\frac{a}{b}\right)^{d} \sum_{\Delta\mathbf{r}} \frac{\rho^{-1}\kappa'^{d-2}}{(\kappa'|\Delta\mathbf{r}|)^{\frac{d-1}{2}}} e^{-\kappa'|\Delta\mathbf{r}|}$$

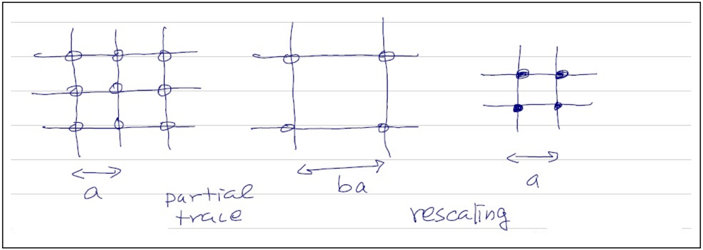
$$\sim \frac{1}{b^d}\int_0^b dr\, r^{d-1} \frac{\rho^{-1}\kappa'^{d-2}}{(\kappa' r)^{\frac{d-1}{2}}} e^{-\kappa' r} \sim \frac{1}{b^d}\frac{1}{\rho\kappa'^2}\int_0^{\kappa' b} dx\, x^{\frac{d-1}{2}} e^{-x}$$

$$\sim \frac{f(\kappa' b)}{\rho\kappa'^2 b^d} \qquad \left(f(x) \sim \begin{cases} x^{\frac{d+1}{2}} & (x \ll 1) \\ f_\infty \text{ (a dimension-less constant)} & (x \gg 1) \end{cases} \right)$$

Supplement: MF estimate of fluctuation (2)



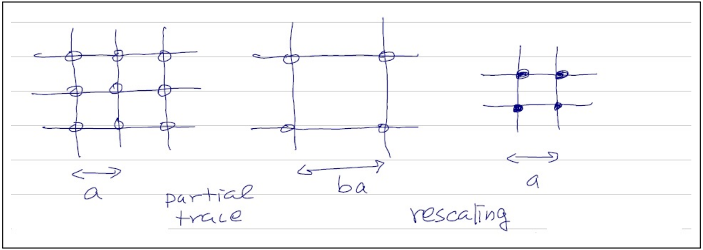
[4-2] General renormalization group (RG) transformation



- In the derivation of the Ginzburg criterion, we introduced the coarse-graining transformation as a Gedankenexperiment.
- The RG transformation consists of two steps: (i) coarse-graining and (ii) rescaling. Schematically,

$$\mathcal{H}_a(S\,|\,\mathbf{K}, L) \xrightarrow{\text{(i)}} \mathcal{H}_{ab}(\tilde{S}\,|\tilde{\mathbf{K}}, L) \xrightarrow{\text{(ii)}} \mathcal{H}_a(\acute{S}\,|\acute{\mathbf{K}}, b^{-1}L)$$

[4-2] General Renormalization Group Transformation



- In the coarse-graining step, we define coarse-grained field and carry out the configuration-space summation of the partition function, with the constraint imposed by the coarse-grained fields.
- In the rescaling step, we redefine the length-scale and the field variables by multiplying them with scaling factors so that the effective Hamiltonian may be the same form as the original one.

Coarse-graining

In the coarse-graining step of the RG procedure, we first define "coarse-grained field", $\tilde{S}_{\mathbf{R}}$, which is defined in terms of $S_{\mathbf{r}}$ in the neighborhood of $\mathbf{R}$, i.e., $\tilde{S}_{\mathbf{R}} = \Sigma(\{S_{\mathbf{r}}\}_{\mathbf{r}\in\Omega_b(\mathbf{R})})$, with some function $\Sigma(\cdots)$. More formally,

$$e^{-\mathcal{H}_a(S|\mathbf{K},L)} \to e^{-\mathcal{H}_{ab}(\tilde{S}|\tilde{\mathbf{K}},L)} \equiv \sum_S \Delta(\tilde{S}\,|\Sigma(S))e^{-\mathcal{H}_a(S|\mathbf{K},L)},$$

where $\mathbf{K}$ is a set of parameters such as $\mathbf{K} \equiv (\beta, H)$.

Example 1 (Ising chain with $b = 3$)

$$\Sigma(S_1, S_2, S_3) = S_2 \qquad \text{(Simple decimation)}$$
$$\Sigma(S_1, S_2, S_3) = (S_1 + S_2 + S_3)/3 \qquad \text{(Local Average)}$$
$$\Sigma(S_1, S_2, S_3) = \mathrm{sign}(S_1 + S_2 + S_3) \qquad \text{(Majority rule)}$$

Example: Coarse-graining of Ising chain ($b = 2$)

- Consider the Ising model of size $L \equiv 2^g$ in one dimension.

$$\mathcal{H}_a(S|\mathbf{K}, L) = -K \sum_{i=0}^{L-1} S_i S_{i+1} - h \sum_{i=0}^{L-1} S_i \qquad (\mathbf{K} \equiv (K, h))$$

- For even L, let us adopt the decimation for the coarse-graining:

$$\tilde{S}_i = S_i \quad (\text{for } i = 0, 2, 4, \cdots, L-2)$$

- Then, $e^{-\mathcal{H}_{2a}(\tilde{S}|\tilde{K},L)} = \sum_{S_1, S_3, \cdots, S_{L-1}} e^{-\mathcal{H}_a(S|K,L)}$. For $h = 0$ we have

$$e^{-\mathcal{H}_{2a}(\tilde{S}|\tilde{K},L)} = \sum_{S_1} e^{K(S_0+S_2)S_1} \sum_{S_3} e^{K(S_2+S_4)S_3} \cdots \sum_{S_{L-1}} e^{K(S_{L-2}+S_0)S_{L-1}}$$

$$\sim e^{\tilde{K} S_0 S_2} e^{\tilde{K} S_2 S_4} \cdots e^{\tilde{K} S_{L-2} S_0} \sim e^{-\mathcal{H}_{2a}(\tilde{S}|\tilde{K},L)} \quad (\tilde{K} \equiv \mathrm{ath}(\mathrm{th}^2 K))$$

Example: Rescaling of Ising chain ($b = 2$)

- Let us use $t \equiv e^{-2K}$ in stead of K for the parameter. Then, the effect of the coarse-graining on t is

$$\tilde{t} = \frac{2t}{1 + t^2}.$$

- The rescaling in the present case is simply

$$\acute{\mathbf{r}} \equiv \mathbf{r}/2, \quad \acute{S}_{\acute{\mathbf{r}}} \equiv \tilde{S}_{\mathbf{r}}, \quad \text{and} \quad \acute{t} \equiv \tilde{t}.$$

- Together with the coarse-graining, we obtain the whole RG transformation,

$$\mathcal{H}_a(S|t, L) \xrightarrow[b=2]{RG} \mathcal{H}_a(\acute{S}|\acute{t}, L/2), \quad \text{with} \quad \acute{t} = \frac{2t}{1 + t^2}.$$

Example: Critical exponent ν

- From the whole RG procedure, we can deduce

$$e^{-r/\xi(t)} \sim \langle S_{\mathbf{r}} S_{\mathbf{0}} \rangle_t = \langle S_{\acute{\mathbf{r}}} S_0 \rangle_{\acute{t}} \sim e^{-\acute{r}/\xi(\acute{t})}$$

- Because $\acute{r} = r/2$,

$$\xi(t) = 2\xi(\acute{t}) \quad \left(\acute{t} = \frac{2t}{1+t^2} \right).$$

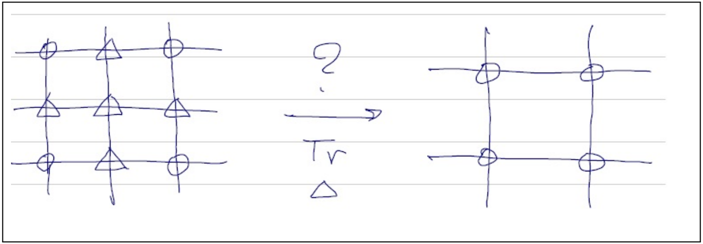
- Since $\acute{t} \approx 2t$ near the fixed-point $t = 0$,

$$\xi(t) \approx 2\xi(2t).$$

- Therefore, for $t \approx 0$,

$$\xi(t) \sim \frac{1}{t} \qquad \Rightarrow \nu = 1 \quad \text{(Exact!)}$$

Can we do the same in 2D case? (1)



Can we do the same in 2D case? (2)

- Coarse-graining by decimation.

$$\tilde{S}_{\mathbf{r}} \equiv S_{\mathbf{r}} \quad \text{for } \mathbf{r} \in \Omega' \equiv \{(2ma, 2na) | m, n = 0, 1, 2, \cdots, L/2\}$$

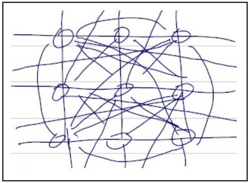
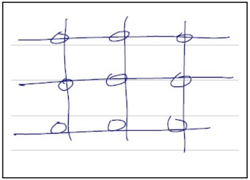
- The partial trace can be taken (at least formally)

$$e^{-\tilde{\mathcal{H}}_{2a}(\tilde{\mathbf{S}}, \tilde{K})} \equiv \operatorname*{Tr}_{\{S_{\mathbf{r}}\}_{\mathbf{r} \in \Omega \setminus \Omega'}} e^{-\mathcal{H}_a(\mathbf{S}, K)}$$

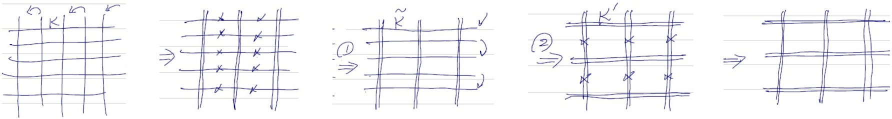
- In general, unlike the 1D case, $\tilde{H}_{2a}$ contains terms other than the two-body nearest-neighbor interactions. For example, it contains the long-range interaction $-K_{\mathbf{r}\mathbf{r}'} S_{\mathbf{r}'} S_{\mathbf{r}}$ where $|\mathbf{r} - \mathbf{r}'| > a$, as well as many-body interactions such as $-K_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3\mathbf{r}_4} S_{\mathbf{r}_1} S_{\mathbf{r}_2} S_{\mathbf{r}_3} S_{\mathbf{r}_4}$.
- Therefore, it is not feasible to study such a model (unless we use machines).

Can we do the same in 2D case? (3)

The renormalized Hamiltonian

is more like  than  .

[4-3] Migdal-Kadanoff approximation for 2D Ising model



- Bunch up two vertical lines.
- Take the partial trace of intermediate spins ($\times$). (Step ①)

$$\mathrm{th}\,\tilde{K} = \mathrm{th}^2\,K$$

- Bunch up two horizontal lines. (Step ②)

$$\acute{K} = 2\tilde{K}$$

- Take the partial trace of intermediate spins ($\times$).

simple Migdal-Kadanoff

$$\acute{t} = \frac{2t^2}{1+t^4} \quad (t \equiv \mathrm{th}\,K,\ \acute{t} \equiv \mathrm{th}\,\acute{K})$$

RG fixed point and $1/\nu$ (general argument)

- Suppose some RG transformation (RGT) yields

$$\acute{t} = R_b(t) \quad (b\text{: the rescaling factor, e.g., } R_2(t) = \tfrac{2t^2}{1+t^4})$$

- We define the RG fixed-point t_c by $t_c = R_b(t_c)$.
- Then, the 'deviation' from the fixed-point changes by RGT as

$$\delta t \to \delta\acute{t} = \acute{t} - t_c = R_b(t) - R_b(t_c) \approx \lambda \delta t \quad (\lambda \equiv R_b'(t_c))$$

- The correlation length after RGT must be smaller than the original by factor b. So, we obtain $\xi(\lambda\delta t) \approx b^{-1}\xi(\delta t)$, which leads to

$$\xi(\delta t) \propto (\delta t)^{-\nu} \text{ where } \lambda^{-\nu} = b^{-1} \quad \text{or} \quad \nu \equiv \frac{\log b}{\log \lambda}\,. \tag{1}$$

RG fixed point and $1/\nu$ (numerical estimates)

- For the Migdal-Kadanoff RGT for 2D Ising model, we have

$$R_2(t_c) = \frac{2t_c^2}{1+t_c^4} = t_c \rightarrow \ t_c = 0.54368\cdots$$

$$(\text{cf: } \ t_c^{\mathrm{exact}} = \sqrt{2} - 1 = 0.4142\cdots)$$

- With some arithmatics, we can get

$$R_2'(t_c) = \frac{2(1-t_c)}{t_c} \approx 1.676$$

$$\rightarrow y_t \equiv 1/\nu \approx \log 1.676 / \log 2 \approx 0.747$$

$$(\text{cf: } \ y_t^{\mathrm{exact}} = 1, \ y_t^{\mathrm{mean\ field}} = 2)$$

Not bad, but ad-hoc (not obvious how to improve).

An improvement of MKRG

- Consider the MKRG step in which b lines, instead of 2, are bunched up at a time. The resulting RG transformation will be

$$\tilde{K} = bK \text{ and } \operatorname{th} \acute{K} = \operatorname{th}^b \tilde{K}, \quad \text{or}$$
$$\operatorname{th} \acute{K} = \operatorname{th}^b (bK)$$

  (The order of bunching and tracing was changed.)
- "bunching-up" two lines to one might be too crude. It may become less harmful if we bunch-up as small number of lines as possible.
- For $b = 1 + \lambda$ ($\lambda \ll 1$), defining $t \equiv \operatorname{th} K$, we obtain

$$\acute{t} = R_b(t) = t + \lambda(1 - t^2) \operatorname{ath} t + \lambda t \log t, \quad \text{or}$$
$$\frac{dR_b}{d \log t} = (1 - t^2) \operatorname{ath} t + t \log t \equiv f(t)$$

Infinitesimal RG (general argument)

- In general, suppose some RG transformation with continuous scaling factor $b = e^{\lambda}$ yields

$$\lim_{\lambda \to 0} \frac{dR(t)}{d\lambda} = f(t).$$

- Obviously, the fixed-point is determined by $t_c = f(t_c)$.
- Starting from the previously obtained expression for $1/\nu$, we get

$$y_t = \frac{1}{\nu} = \frac{\log\left(\frac{dR_b}{dt}\right)_{t_c}}{\log b} = \frac{1}{\lambda}\left(\frac{dR_{1+\lambda}}{dt} - 1\right)$$
$$= \frac{d}{dt}\left(\frac{R_{1+\lambda} - t}{\lambda}\right) = \frac{d}{dt}\left(\frac{R_{1+\lambda} - R_1}{\lambda}\right) = \left(\frac{d}{dt} f(t)\right)_{t_c}$$

Infinitesimal MKRG (numerical estimates)

- From $t_c = f(t_c) = (1 - t_c^2)\,\mathrm{ath}\, t_c + t_c \log t_c$,, we obtain

$$t_c = \sqrt{2} - 1 = t_c^{\mathrm{exact}} \quad !$$

- As for y_t, we have

$$y_t = f'(t_c) = 0.7535 \cdots ,$$

slightly closer to $y_t^{\mathrm{exact}} = 1$ than the simple MKRG with $b = 2$.

Better, but still ad-hoc (not obvious how to further improve).

Exercise

- By solving the 1D Ising model, compute the correlation function $G(r) \equiv \langle S_r S_0 \rangle$ and the correlation length ξ. Verify $\xi \propto t^{-1}$.

hint:

$$\langle S_r S_0 \rangle = \mathrm{Tr}\left(T^{L-r} \sigma T^r \sigma\right) \Big/ \mathrm{Tr}\left(T^L\right)$$

where

$$T_{S'S} \equiv e^{KS'S} \quad (2 \times 2 \text{ matrix})$$

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$